Calculus II: Spring 2018

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Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 6.1, 6.2*, 6.3*.
- *Calculus*, Spivak, 3rd Ed.: Section 18.

Keywords: chain rule, natural logarithm

The Natural Logarithm I

Today we introduce the *natural logarithm* function as the inverse function of $\exp(x)$.

Let $f(x)$ be a one-to-one function with domain A and range B. Then, its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \qquad \Leftrightarrow \qquad y = f(x).$$

In words,

> If $y = f(x)$ is an output of f then y is an input of f^{-1} and $f^{-1}(y) = x$.

Example:

1. Let $f(x) = 1 - \frac{1}{x}$ with domain A being the collection of all real numbers $x > 0$. The range of f, B, is the collection of all real numbers $y < 1$. The function $f(x)$ is one-to-one and its inverse function is
$$f^{-1}(y) = \frac{1}{1-y}, \quad y < 1.$$
2. Let $f(x) = 2x^2 - 1$, $x \geq 0$. Then, $f^{-1}(y) = \sqrt{\frac{y+1}{2}}$, $y \geq -1$.

Important Remarks:

1. To determine the inverse function $f^{-1}(y)$ of a one-to-one function $f(x)$, solve the equation
$$y = f(x)$$
for x in terms of y.
2. It is important to remember that $f^{-1}(y) \neq \frac{1}{f(y)}$, in general. For example, if $f(x) = 1 - \frac{1}{x}$ then
$$\frac{1}{f(y)} = \frac{y}{y-1} \neq \frac{1}{1-y} = f^{-1}(y)$$

3. Let $f(x)$ be a one-to-one function with domain A and range B. Then, f and its inverse function f^{-1} satisfy the following functional relationship:

$$f(f^{-1}(y)) = y, \quad \text{for every } y \text{ in } B,$$
$$f^{-1}(f(x)) = x, \quad \text{for every } x \text{ in } A.$$

The derivative of inverse functions

Let $f(x)$ be a one-to-one function with domain A and range B. Then, f and its inverse function f^{-1} satisfy the following functional relationship:

$$f(f^{-1}(y)) = y, \quad \text{for every } y \text{ in } B, \qquad (*)$$
$$f^{-1}(f(x)) = x, \quad \text{for every } x \text{ in } A.$$

If $f(x)$ is also a differentiable function (i.e. the derivative $f'(x)$ exists for every x in A) then its inverse function is also differentiable. In fact, the derivative of $f^{-1}(y)$ can be determined in terms of the derivative of $f'(x)$.

First, we recall some results from Calculus I.

Check your understanding

Compute $\frac{dy}{dx}$ where

$$y = x^2 + 4x + \frac{1}{x^2 + 4x}$$

Chain Rule

Let f and g be differentiable functions. Then,

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

Example: Let $f(x) = x + \frac{1}{x}$ and $g(x) = x^2 + 4x$. We have

$$f'(x) = 1 - \frac{1}{x^2}, \qquad g'(x) = 2x + 4.$$

Then, the chain rule states that

$$\frac{d}{dx}\left(x^2 + 4x + \frac{1}{x^2 + 4x}\right) = \frac{d}{dx} f(g(x)) = f'(x^2 + 4x) \cdot g'(x) = \left(1 - \frac{1}{(x^2 + 4x)^2}\right)(2x + 4)$$

You can check that this agrees with your calculation above.

Check your understanding

Recall the function $f(x) = 1 - \frac{1}{x}$ from October 11 Lecture. We determined the inverse function to be

$$f^{-1}(y) = \frac{1}{1-y}$$

1. Compute

$$\frac{d}{dy} f^{-1}(y).$$

2. Compute $f'(x)$.

3. Show that

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{f'(f^{-1}(y))}.$$

Derivative of the inverse function

Let $f(x)$ be a differentiable one-to-one function. Suppose that $f'(f^{-1}(y)) \neq 0$, for all y. Then, $f^{-1}(y)$ is differentiable and

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{f'(f^{-1}(y))}$$

Proof: This follows from the functional relationship $(*)$ and the chain rule. We have, for every y,

$$f(f^{-1}(y)) = y.$$

Now, differentiating with respect to y (remember, we are wanting the derivative of the function $f^{-1}(y)$ with respect to y) and using the chain rule, we find

$$1 = \frac{d}{dy}\left(f(f^{-1}(y))\right) = f'(f^{-1}(y)) \cdot (f^{-1})'(y)$$

$$\Longrightarrow \quad \frac{d}{dy}\left(f^{-1}(y)\right) = (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$

The natural logarithm

Recall the following facts about $\exp(x)$:

1. $\exp(x)$ is strictly increasing. Hence, $\exp(x)$ is one-to-one.
2. The domain of $\exp(x)$ is the collection of all real numbers.
3. The range of $\exp(x)$ is the collection of all $y > 0$.

Hence, $\exp(x)$ **has an inverse function** $\exp^{-1}(y)$.

Remark:

- the domain of $\exp^{-1}(y)$ is the collection of all $y > 0$
- the range of $\exp^{-1}(y)$ is the collection of all real numbers

We will often write $\exp^{-1}(x)$ instead of $\exp^{-1}(y)$. Remember, it doesn't matter what symbol we use for our input variable as long as we are consistent.

Now, since $f(x) = \exp(x)$ is a differentiable function so is $\exp^{-1}(y)$. Thus, using the formula for the derivative of the inverse function

$$\frac{d}{dy}\exp^{-1}(y) = \frac{1}{f'(\exp^{-1}(y))}$$

Recall that $f'(x) = \exp(x)$. Therefore, $f'(\exp^{-1}(y)) = \exp(\exp^{-1}(y)) = y$, using functional property $(*)$. Hence,

$$\frac{d}{dy}\exp^{-1}(y) = \frac{1}{y} \qquad (**)$$

A Fundamental Interlude

Let $f(x)$ be a function. An **antiderivative** of $f(x)$ is a differentiable function $F(x)$ satisfying

$$\frac{d}{dx}F(x) = f(x).$$

Proposition: *If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ then*

$$F(x) = G(x) + c,$$

for some constant c.

The most important Theorem you saw in Calculus I was an approach to determining the antiderivative of a continuous function.

Fundamental Theorem of Calculus

Let $f(x)$ be a continuous function defined on the closed interval $a \leq x \leq b$. Then, the function

$$F(x) = \int_a^x f(u)du$$

is an antiderivative of $f(x)$.

The natural logarithm II

We can restate $(**)$ as follows:

$\exp^{-1}(x)$ *is an antiderivative of* $g(x) = \frac{1}{x}$.

We define the **natural logarithm function** to be the function

$$\ln(x) = \int_1^x \frac{dt}{t} \quad x > 0$$

By the Fundamental Theorem of Calculus, $\ln(x)$ is an antiderivative of $g(x) = \frac{1}{x}$. Hence, the Proposition implies that there is a constant c so that

$$\exp^{-1}(x) = \ln(x) + c.$$

Since $\exp(0) = 1$, we have $\exp^{-1}(1) = 0$. Hence,

$$0 = \exp^{-1}(1) = \log(1) + c = \int_1^1 \frac{dt}{t} + c = c.$$

Hence,

The **natural logarithm function** $\ln(x)$ defined above *is* the inverse function $\exp^{-1}(x)$,

$$\ln(x) = \exp^{-1}(x)$$