Contact: `gmelvin@middlebury.edu`

March 1 Lecture

Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.4.
- *Calculus*, Spivak, 3rd Ed.: Section 22.
- *AP Calculus BC*, Khan Academy: Comparison Tests.

Keywords: *Direct Comparison Test, p-series Test, Limit Comparison Test.*

Convergence Tests for Series III

Today we consider comparison tests for series. These are tests for convergence of a series of positive terms.

At the end of yesterday's lecture we investigated the behaviour of the series

$$\sum_{n=1}^{\infty} \frac{1}{5^n + 3}$$

We determined the following:

- The sequence of partial sums (s_m) associated to $\sum_{n=1}^{\infty} \frac{1}{5^n+3}$ is *increasing*. Therefore, to show convergence of (s_m) it suffices, by the Monotonic Bounded Theorem, to show that (s_m) is *bounded above*. We do this by comparing $\sum_{n=1}^{\infty} \frac{1}{5^n+3}$ with known behaviour of another series.
- The sequence of partial sums (t_m) associated to the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{4}$ are increasing and satisfy $t_m \leq \frac{1}{4}$, for $m = 1, 2, 3, \ldots$.
- Since, for each $n = 1, 2, 3, \ldots$,

  $$5^n < 5^n + 3 \quad \Longrightarrow \quad \frac{1}{5^n + 3} < \frac{1}{5^n}$$

  we have

  $$s_m = \sum_{n=1}^{m} \frac{1}{5^n + 3} < t_m = \sum_{n=1}^{m} \frac{1}{5^n} \leq \frac{1}{4},$$

  for every $m = 1, 2, 3, \ldots$.

  Hence, (s_m) is ____________________.
- Therefore, by the Montonic Bounded Theorem the sequence (s_m) is ____________________; hence, the series $\sum_{n=1}^{\infty} \frac{1}{5^n+3}$ is ____________________.

What you have shown in the previous exercise is the idea underlying the proof of the following result.

Direct Comparison Test (DCT)

Let $\sum a_n$ and $\sum b_n$ be series having positive terms.

1) Suppose that, for each n, $a_n \leq b_n$, and $\sum b_n$ is convergent. Then, $\sum a_n$ is convergent.

2) Suppose that, for each n, $a_n \geq b_n$, and $\sum b_n$ is divergent. Then, $\sum a_n$ is divergent.

Remark: To be able to use the Direct Comparison Test effectively, we need to have a bank of series **whose behaviour is known** that we can use for comparison purposes. At the moment, we know the behaviour of the following types of series: *geometric series*, *telescoping series*, *the Harmonic Series*.

We record the following facts:

- in Homework you will show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent;
- we've seen that the Harmonic Series $\sum \frac{1}{n}$ is divergent.

p-series Test

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p is a real number. Then,

1. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p \geq 2$.
2. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent if $p \leq 1$.

Proof:

1. If $p \geq 2$ then, for each $n = 1, 2, 3, \ldots$,

$$n^p \geq n^2 \quad \Longrightarrow \quad 0 < \frac{1}{n^p} \leq \frac{1}{n^2}$$

Hence, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, we can apply DCT to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is also convergent.

2. If $p \leq 1$ then, for each $n = 1, 2, 3, \ldots$,

$$n^p \leq n \quad \Longrightarrow \quad \frac{1}{n} \leq \frac{1}{n^p}$$

Hence, since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, we can apply DCT to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is also divergent.

Remark: Although we can't show it yet, it is a fact (**which you are allowed to assume**) that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent whenever $p > 1$.

Example:

1. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^3+n+1}$. We are going to compare this series with the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

   For $n = 1, 2, 3, \ldots$, we have

$$n^3 < n^3 + n + 1 \qquad \Longrightarrow \qquad \frac{1}{n^3+n+1} < \frac{1}{n^3}$$

   Since the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent, by the p-series test, the series $\sum_{n=1}^{\infty} \frac{1}{n^3+n+1}$ is convergent, by DCT.

2. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$$

   We rewrite the summand $\frac{1}{2n-1} = \frac{1}{2}\left(\frac{1}{n-1/2}\right)$. Now, for each $n = 1, 2, 3, \ldots$

$$n - 1/2 < n \qquad \Longrightarrow \qquad \frac{1}{n} < \frac{1}{n-1/2}$$

   Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (p-series with $p = 1$), the series $\sum_{n=1}^{\infty} \frac{1}{n-1/2}$ is divergent, by DCT. Hence, the series $\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n-1/2} = \sum_{n=1}^{\infty} \frac{1}{2n-1}$ is divergent.

Check your understanding

1. Use the Direct Comparison Test to determine whether the series $\sum_{n=1}^{\infty} \frac{5}{3n^5+3n^2+n}$ converges or diverges.

2. Explain why you can't use the Direct Comparison Test to compare the behaviour of the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ with the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

The series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ '*looks like*' the convergent series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ and it seems entirely reasonable to expect that the convergence behaviour of these series should coincide. As you've just seen, we can't make use of the Direct Comparison Test to compare these series. However, all is not lost:

Observation: in a few lectures we will learn the technique of *mathematical induction*. This will allow us to show the following

$$\frac{1}{2^n - 1} \leq \frac{3}{2^n}, \quad \text{for } n = 1, 2, 3, \ldots$$

Check your understanding
Use the observation to show that the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ is convergent.

The following test for convergence formalises this notion of comparing series with unknown convergence behaviour with *similar looking* convergent series.

Limit Comparison Test (LCT)

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If the sequence $\left(\frac{a_n}{b_n}\right)$ is convergent and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0,$$

then either both series converge or both series diverge.

Example:

1. Consider the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$. As n gets very large the terms of the series begin to *look like* $\frac{1}{\sqrt{n}}$ (i.e. for n very large, $\sqrt{n+2}$ is approximately $\sqrt{n}$).

   The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2}$, and therefore divergent. It seems reasonable to expect that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ is also divergent. However, since

   $$\_\_\_\_\_\_\_\_\_\_ < \_\_\_\_\_\_\_\_\_\_, \quad \text{for } n = 1, 2, 3, \ldots$$

   we can't apply the ____________________ to show that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ is divergent.

   However, if we let $b_n = \frac{1}{\sqrt{n+2}}$ and $a_n = \frac{1}{\sqrt{n}}$ then

   $$\frac{a_n}{b_n} = \frac{\sqrt{n+2}}{\sqrt{n}} = \frac{\sqrt{n(1+2/n)}}{\sqrt{n}} = \sqrt{1+2/n} \longrightarrow \sqrt{1+0} = 1, \quad \text{as } n \to \infty$$

   Hence, since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$ and the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent (p-series with $p = \frac{1}{2}$), the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ is also divergent, by the Limit Comparison Test.

2. Let $a_n = \frac{1}{2^n}$, $b_n = \frac{1}{2^n - 1}$. Then,

   $$\frac{a_n}{b_n} = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n} \longrightarrow 1 - 0 = 1, \quad \text{as } n \to \infty.$$

   Hence, since $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$ and the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent (Geometric Series Theorem), the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ is also convergent, by the Limit Comparison Test.

Flex those mathematical muscles!

Use the Limit Comparison Test to determine whether the following series converge or diverge.

1.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$$

2.

$$\sum_{n=1}^{\infty} \frac{n-1}{n^2\sqrt{n}}$$

3.

$$\sum_{n=1}^{\infty} \frac{9^n}{10^n-4}$$

Appendix: In this Appendix we provide a proof of the Limit Comparison Test.

Let (a_n) and (b_n) be sequences of positive terms. Suppose that $\left(\frac{a_n}{b_n}\right)$ is convergent and $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$. Hence, for $\epsilon = \frac{c}{2}$ we can find N so that

$$n \geq N \qquad \Longrightarrow \qquad \left|\frac{a_n}{b_n} - c\right| < c_2$$

That is, for $n \geq N$ we have

$$\frac{c}{2} = c - \frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2} = c + \frac{c}{2}$$

Hence, since $b_n > 0$, we can rewrite this inequality as

$$\frac{c}{2} b_n < a_n < \frac{3c}{2} b_n$$

Hence,

- if $\sum_{n=N}^{\infty} b_n$ is divergent so is $\sum_{n=N}^{\infty} a_n$, using the left inequality and DCT. Hence, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n$ is also divergent.
- if $\sum_{n=N}^{\infty} b_n$ is convergent then so is $\sum_{n=N}^{\infty} a_n$, using the right inequality and DCT. Hence, $\sum_{n=1}^{\infty} a_n \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n$ is also convergent.