Middlebury College

Calculus II: Spring 2018

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Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 6.1, 6.2*, 6.3*.
- *Calculus*, Spivak, 3rd Ed.: Section 18.

Keywords: one-to-one functions, inverse functions

Inverse Functions

Today we introduce the notion of an *inverse function.* We will see that the fact that $\exp(x)$ is *strictly increasing* means the equation $c = \exp(x)$, where $c > 0$ is a constant, has a unique solution. Determining this unique solution will require us to recall, in our next lecture, the *Fundamental Theorem of Calculus.*

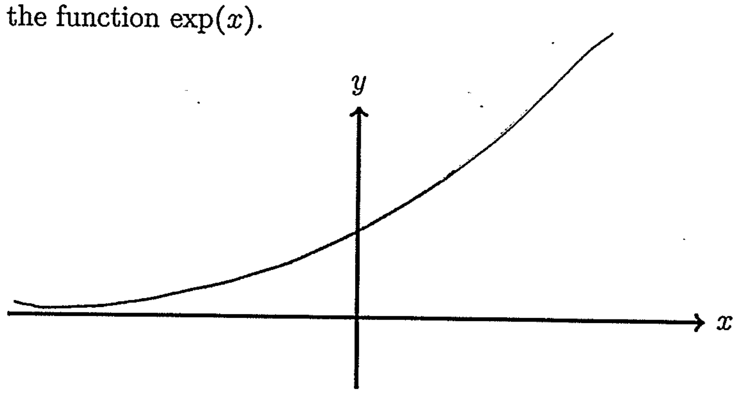
Recall: for $x > 0$ the exponential function $\exp(x)$ satisfies

$$\exp(x) > 1 + x$$

Hence, $\exp(x)$ is unbounded.

Check your understanding

Draw the graph of the function $\exp(x)$.



Note: graph is continuous because exp(x) differentiable

Let $c > 0$ be your favourite positive real number. Explain why the equation

$$c = \exp(x)$$

has a unique solution. Since exp(x) is continuous and $0 < \exp(x) < +\infty$ attains all values, the equation $c = \exp(x)$ has a solution. This solution is unique because exp(x) strict inc.

Let $f(x)$ be a function. We say that $f(x)$ is one-to-one if $f(x) \neq f(y)$ whenever $x \neq y$ lie in the domain of f. In words,

$f(x)$ is one-to-one if no two distinct inputs give the same output.

Example:

1. The function $f(x) = x^2$, x any real number, is not one-to-one because $f(-5) = f(5)$.
2. The function $f(x) = \frac{2}{1-x}$, $x > 1$, is one-to-one: if there exists x, y such that

$$f(x) = f(y) \quad \Longrightarrow \quad \frac{2}{1-x} = \frac{2}{1-y} \quad \Longrightarrow \quad 1-x = 1-y \quad \Longrightarrow \quad x = y$$

3. The function $g(x) = x^2$, $x \geq 0$, is one-to-one: if there exists x, y such that $g(x) = g(y)$ then

$$x^2 = y^2 \quad \Longrightarrow \quad (x-y)(x+y)=0 \quad \Longrightarrow \quad x = \pm y$$

Since $x, y \geq 0$ we must have $x = y$.

4. The function $h(x) = x^3$, x any real number, is one-to-one: first you can check that

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2).$$

If $x \neq y$ then the right hand side of the above expression can only be equal to 0 whenever

$$x^2 + xy + y^2 = 0.$$

By completing the square we find

$$x^2 + xy + y^2 = \left(x + \frac{y}{2}\right)^2 + \frac{3}{4}y^2 \geq 0$$

Thus,

$$0 = x^2 + xy + y^2 \quad \Leftrightarrow \quad x + \frac{y}{2} = 0 \text{ and } y = 0 \quad \Leftrightarrow \quad x = y = 0,$$

contradicting our assumption that $x \neq y$. Hence, $x^2 + xy + y^2 > 0$ whenever $x \neq y$, so that $x^3 \neq y^3$ whenever $x \neq y$.

We have the following useful test to determine when a function is one-to-one:

Horizontal Line Test

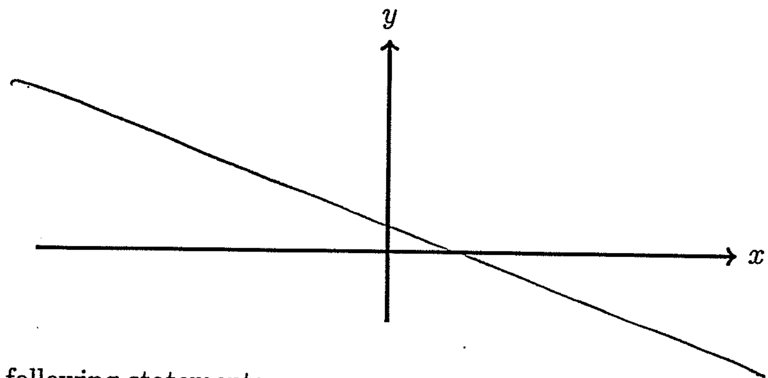
$f(x)$ is one-to-one if every horizontal line intersects its graph at most once.

Check your understanding

1. Let $f(x) = \exp(x)$ be the exponential function. Explain why $\exp(x)$ is one-to-one.

exp(x) is strictly increasing

2. Suppose that $f(x)$ is a strictly decreasing function. Draw a representative graph of the function below



3. Complete the following statements:

- If a function $f(x)$ is strictly decreasing then $f(x)$ is one-to-one.
- If a function $f(x)$ is strictly increasing then $f(x)$ is one-to-one.

Inverse functions

In this paragraph we will introduce the notion of an inverse function. You have determined above the following result.

> Let $f(x)$ be a strictly increasing/decreasing function and let c be a real number. Then, the equation
> $$c = f(x)$$
> has *at most one solution*.

Example:

1. Let $f(x) = x^3$, with domain being the collection of all real numbers x, and $c = -27$. Then, $x = -3$ is a solution to the equation $-27 = x^3$. Moreover, this is the only solution.
2. Let $f(x) = 1 - \frac{1}{x}$, with domain being the collection of real numbers $x > 0$. Let $c = -1$. Then, the equation
$$-1 = 1 - \frac{1}{x}$$
has the unique solution $x =$ 1/2.
If $c = 3$ then there is no solution, $x > 0$.

Definition: Let $f(x)$ be a function. The **range of** $f(x)$ is the collection of all outputs of $f(x)$.
Example:

1. Let $f(x) = x^3$ with domain being the collection of all real numbers x. Then, the range of $f(x)$ is the collection of all real numbers: if c is any real number then $c = \left(\sqrt[3]{c}\right)^3 = f\left(\sqrt[3]{c}\right)$. That is, c is the output of some input for the function $f(x) = x^3$.

2. Let $f(x) = 1 - \frac{1}{x}$ with domain being the collection of all real numbers $x > 0$. Then, the range of $f(x)$ is the collection of all real numbers $c < 1$: if $c < 1$ then $x = \frac{1}{1-c} > 0$ satisfies $f(x) = c$.

3. Let $f(x) = 2x^2 - 1$, $x \geq 0$. Then, the range of f is the collection of real numbers $c \geq -1$: if $c \geq -1$ then $x = \sqrt{\frac{c+1}{2}}$ satisfies $f(x) = c$.

We are now ready to introduce our main definition: let $f(x)$ be a one-to-one function with domain A and range B. Then, its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \qquad \Leftrightarrow \qquad y = f(x).$$

In words,

> If $y = f(x)$ is an output of f then y is an input of f^{-1} and $f^{-1}(y) = x$.

Example:

1. Let $f(x) = 1 - \frac{1}{x}$ with domain A being the collection of all real numbers $x > 0$. The range of f, B, is the collection of all real numbers $y < 1$. The function $f(x)$ is one-to-one and its inverse function is
$$f^{-1}(y) = \frac{1}{1-y}, \quad y < 1.$$

2. Let $f(x) = 2x^2 - 1$, $x \geq 0$. Then, $f^{-1}(y) = \sqrt{\frac{y+1}{2}}$, $y \geq -1$

Important Remarks:

1. To determine the inverse function f^{-1} of a one-to-one function f, solve the equation
$$y = f(x)$$
for x in terms of y.

2. It is important to remember that $f^{-1}(y) \neq \frac{1}{f(y)}$, in general. For example, if $f(x) = 1 - \frac{1}{x}$ then
$$\frac{1}{f(y)} = \frac{y}{y-1} \neq \frac{1}{1-y} = f^{-1}(y)$$

3. Let $f(x)$ be a one-to-one function with domain A and range B. Then, f and its inverse function f^{-1} satisfy the following functional relationship:
$$f(f^{-1}(y)) = y, \quad \text{for every } y \text{ in } B,$$
$$f^{-1}(f(x)) = x, \quad \text{for every } x \text{ in } A.$$