Middlebury College

Calculus II: Spring 2018

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March 16 Lecture

Keywords: the exponential function

An exp-traordinary function II

Recall that

$$\exp(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} = \lim_{m\to\infty} s_m(x).$$

Remark: I claimed yesterday that $\exp(x) = e^x$, where $e = \exp(1) = 2.71828...$ is Euler's number. You should not take anything that I (or anyone else) say(s) on blind faith. We are going to recover a lot of the known properties of e^x directly by analysing the function $\exp(x)$, which should hopefully convince you that my claim has some merit.

We obtained the following

Remarkable Property

$$\exp(x) \cdot \exp(y) = \underline{\exp(x+y)} \qquad (*)$$

Property $(*)$ is similar to an *exponent law* and has lots of remarkable consequences. For example, suppose that x is any positive real number. Then,

$$\begin{aligned} 1 &= \exp(0) \\ &= \exp(x + (-x)) \\ &= \exp(x) \cdot \exp(-x) \end{aligned}$$

In particular,

- $\exp(-x) = \frac{1}{\exp(x)}$, for any real number x.
- $\exp(x) \neq 0$, for any real number x.

Check your understanding

1. Use $1 = \exp(x)\exp(-x)$ and the fact that $\exp(x) > 1$, whenever $x > 0$, to deduce that $\exp(x) > 0$, for all x.

$$1 = \underbrace{\exp(x)}_{>0} \cdot \exp(-x)$$

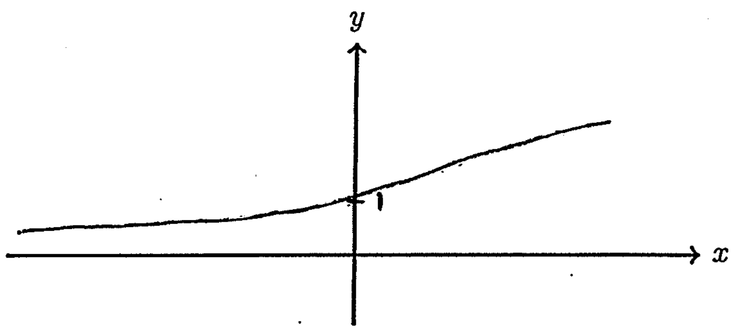
$$\Rightarrow \quad \exp(-x) > 0$$

2. Let $x < y$ and write $y = x + h$, where $h > 0$. Use (*) to show that $\exp(y) > \exp(x)$. (*Hint: recall that* $\exp(h) > 1$ *whenever* $h > 0$)

$$
\begin{aligned}
\exp(y) &= \exp(x+h) \\
&= \exp(x)\cdot\exp(h) \\
&> \exp(x)\cdot 1 \quad , \text{ since } \exp(h) > 1 \\
&= \exp(x)
\end{aligned}
$$

Hence, the exponential function is **strictly increasing.**

3. Based on your investigations, draw the graph of the function $\exp(x)$.



Not sure whether exp(x) is bounded though.

O Calculus, Where Art Thou?

Let h be a real number and consider the series

$$\frac{\exp(h)-1}{h} = 1 + \frac{h}{2!} + \frac{h^2}{3!} + \ldots = 1 + \sum_{n=1}^{\infty} \frac{h^n}{(n+1)!}.$$

Using the Ratio Test it can be shown that this series is (absolutely) convergent for any h.

CHECK YOUR UNDERSTANDING

1. As h gets close, but not equal,to 0, describe what happens to the expression

$$\frac{\exp(h)-1}{h}$$

gets close to 1

2. Complete the following statement

$$\boxed{\lim_{h\to 0} \frac{\exp(h)-1}{h} = \underline{\quad 1 \quad}}$$

Recall what it means for a function $f(x)$ to be *differentiable at* $x = a$: we say that $f(x)$ is **differentiable at** $x = a$ if the following limit exists

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

In this case we write

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

If $f(x)$ is differentiable for every input value x, then we define the **derivative of** $f(x)$ to be the function

$$\boxed{f'(x) \stackrel{def}{=} \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

Let a be a real number. Using the **Remarkable Property**, we find

$$\frac{\exp(a+h)-\exp(a)}{h} = \exp(a)\left[\frac{\exp(h)-1}{h}\right]$$

Hence,

$$\exp'(a) = \lim_{h\to 0}\frac{\exp(a+h)-\exp(a)}{h} = \exp(a)$$

Hence,

The function $\exp(x)$ is differentiable and

$$\frac{d}{dx}\exp(x) = \exp(x)$$

You have seen, and determined most of, the following properties of $\exp(x)$:

- $\exp(x) > 1$, for any real number $x > 0$.
- $\exp(0) = 1$.
- $\exp(-x) = \frac{1}{\exp(x)}$, for any real number x.
- $\exp(x) > 0$, for any real number x.
- $\exp(x+y) = \exp(x)\cdot\exp(y)$, for any real numbers x, y. (*)
- $\exp(x)$ is **strictly increasing.**
- $\exp(x) = e^x$, where $e = \exp(1)$ is **Euler's number.**
- $\exp(x)$ is differentiable, for every x, and its derivative is itself

$$\frac{d}{dx}\exp(x) = \exp(x).$$