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# March 16 Lecture

Keywords: the exponential function

## An exp-traordinary function II

Recall that

$$\exp(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} = \lim_{m \to \infty} s_m(x).$$

**Remark:** I claimed yesterday that $\exp(x) = e^x$, where $e = \exp(1) = 2.71828...$ is Euler's number. You should not take anything that I (or anyone else) say(s) on blind faith. We are going to recover a lot of the known properties of $e^x$ directly by analysing the function $\exp(x)$, which should hopefully convince you that my claim has some merit.

We obtained the following

**Remarkable Property**

$$\exp(x) \cdot \exp(y) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \qquad (*)$$

Property $(*)$ is similar to an *exponent law* and has lots of remarkable consequences. For example, suppose that $x$ is any positive real number. Then,

$$\begin{aligned} 1 &= \exp(0) \\ &= \exp(x + (-x)) \\ &= \exp(x) \cdot \exp(-x) \end{aligned}$$

In particular,

- $\exp(-x) = \frac{1}{\exp(x)}$, for any real number $x$.
- $\exp(x) \neq 0$, for any real number $x$.

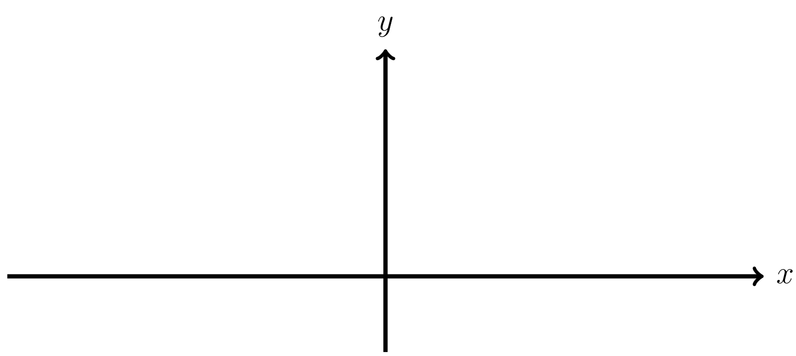
### Check your understanding

1. Use $1 = \exp(x)\exp(-x)$ and the fact that $\exp(x) > 1$, whenever $x > 0$, to deduce that $\exp(x) > 0$, for all $x$.

2. Let $x < y$ and write $y = x + h$, where $h > 0$. Use $(*)$ to show that $\exp(y) > \exp(x)$. (*Hint: recall that* $\exp(h) > 1$ *whenever* $h > 0$)

Hence, the exponential function is **strictly increasing**.

3. Based on your investigations, draw the graph of the function $\exp(x)$.




## O Calculus, Where Art Thou?

Let $h$ be a real number and consider the series

$$\frac{\exp(h) - 1}{h} = 1 + \frac{h}{2!} + \frac{h^2}{3!} + \ldots = 1 + \sum_{n=1}^{\infty} \frac{h^n}{(n+1)!}.$$

Using the Ratio Test it can be shown that this series is (absolutely) convergent for any $h$.

Check your understanding

1. As $h$ gets close, but not equal,to 0, describe what happens to the expression

$$\frac{\exp(h) - 1}{h}$$

2. Complete the following statement

$$\lim_{h \to 0} \frac{\exp(h) - 1}{h} = \_\_\_\_\_\_\_\_\_\_$$

Recall what it means for a function $f(x)$ to be *differentiable at* $x = a$: we say that $f(x)$ is **differentiable at** $x = a$ if the following limit exists

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

In this case we write

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

If $f(x)$ is differentiable for every input value $x$, then we define the **derivative of** $f(x)$ to be the function

$$\boxed{f'(x) \overset{def}{=} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}$$

Let $a$ be a real number. Using the **Remarkable Property**, we find

$$\frac{\exp(a+h) - \exp(a)}{h} = \underline{\qquad\qquad\qquad\qquad\qquad}$$

Hence,

$$\exp'(a) = \lim_{h\to 0} \frac{\exp(a+h) - \exp(a)}{h} = \underline{\qquad\qquad\qquad}$$

Hence,

> The function $\exp(x)$ is ____________________ and
>
> $$\frac{d}{dx}\exp(x) = \underline{\qquad\qquad}$$

You have seen, and determined most of, the following properties of $\exp(x)$:

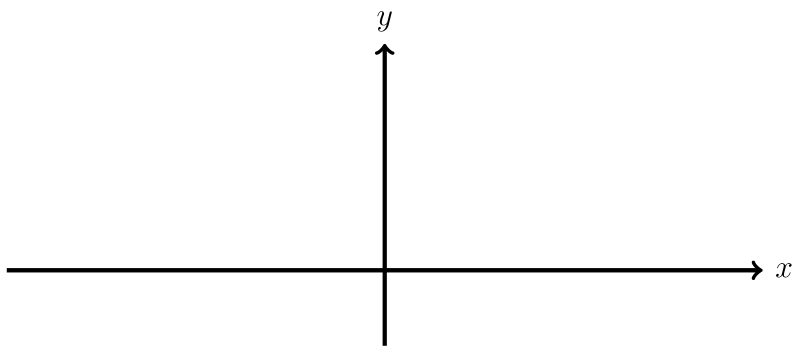
- $\exp(x) > 1$, for any real number $x > 0$.
- $\exp(0) = 1$.
- $\exp(-x) = \frac{1}{\exp(x)}$, for any real number $x$.
- $\exp(x) > 0$, for any real number $x$.
- $\exp(x + y) = \exp(x) \cdot \exp(y)$, for any real numbers $x, y$. $(*)$
- $\exp(x)$ is **strictly increasing**.
- $\exp(x) = e^x$, where $e = \exp(1)$ is **Euler's number**.
- $\exp(x)$ is differentiable, for every $x$, and its derivative is itself

$$\frac{d}{dx}\exp(x) = \exp(x).$$

**Remark:** compare these properties with what you know about $e^x$, are there any additional properties that we've yet to show?

CHECK YOUR UNDERSTANDING

Draw the graph of the function $\exp(x)$.




Let $c > 0$ be your favourite positive real number. Explain why the equation

$$c = \exp(x)$$

has a unique solution.