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HW MARCH 16 : MATH 122B

1a) Recall: A natural number m is divisible by 3 if $m = 3p$, for some p.

$P(n)$: $n^3 - n$ is divisible by 3.

i.e $P(n)$: $n^3 - n = 3p$, for some p.

Base case:

$P(1)$: $1^3 - 1 = 0 = 3.0$ ✓

Inductive Step

Assume $P(k)$: $k^3 - k = 3p$, some p.

Want to show

$P(k+1)$: $(k+1)^3 - (k+1) = 3q$, some q.

Now: $(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1$

$= k^3 - k + 3k^2 + 3k$

$= 3p + 3k^2 + 3k$, by inductive hyp.

$= 3(p + k^2 + k)$

$\Rightarrow$ $P(k+1)$.

Hence, $P(n)$, for all n, by mathematical induction.

b) Let $P(n): \sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$

Base case:

$P(1): \sum_{k=1}^{n} k = 1 = \frac{1}{2}1.(1+1)$ ✓

Inductive step:

Assume $P(r): \sum_{k=1}^{r} k = \frac{1}{2}r(r+1)$

Want to show: $P(r+1): \sum_{k=1}^{r+1} k = \frac{1}{2}(r+1)(r+2)$

Now: $\sum_{k=1}^{r+1} k = \sum_{k=1}^{r} k + (r+1)$

$= \frac{1}{2}r(r+1) + (r+1)$

$= \frac{(r+1)}{2}\left[r + 2\right]$

$= \frac{1}{2}(r+1)(r+2)$ ✓

Hence, $P(r+1)$

$\Rightarrow$ $P(n)$, for all n, by mathematical induction.

c) Let $P(n): \sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$

<u>Base case</u>

$P(1): \sum_{k=1}^{1} k^2 = 1^2 = 1 = \frac{1}{6} 1 . (1+1)(2.1+1)$ ✓

<u>Inductive step</u>:

Assume $P(r): \sum_{k=1}^{r} k^2 = \frac{1}{6} r(r+1)(2r+1)$

Want to show:

$P(r+1): \sum_{k=1}^{r+1} k^2 = \frac{1}{6}(r+1)(r+2)(2(r+1)+1)$

Now:

$$\sum_{k=1}^{r+1} k^2 = \sum_{k=1}^{r} k^2 + (r+1)^2$$

$$= \frac{1}{6}(r+1)(r+2)(2r+1) + (r+1)^2$$

$$= \frac{(r+1)}{6}\left[(r+2)(2r+1) + 6(r+1)\right]$$

$$= \frac{r+1}{6}\left[2r^2 + 11r + 8\right]$$

$$= \frac{(r+1)}{6}\left[2(r+2)\right.$$

c) Let $P(n) : \sum_{k=1}^{n} k^2 = \frac{1}{6} n (n+1)(2n+1)$

<u>Base case</u>

$P(1) : \sum_{k=1}^{1} k^2 = 1^2 = 1 = \frac{1}{6} \cdot 1 \cdot (1+1)(2 \cdot 1 + 1)$

<u>Inductive Step</u>:

Assume $P(r) : \sum_{k=1}^{r} k^2 = \frac{1}{6} r (r+1)(2r+1)$

WTS: $P(r+1) : \sum_{k=1}^{r+1} k^2 = \frac{1}{6}(r+1)(r+2)(2(r+1)+1)$

Now,

$$\sum_{k=1}^{r+1} k^2 = \sum_{k=1}^{r} k^2 + (r+1)^2$$

$$= \frac{1}{6} r (r+1)(2r+1) + (r+1)^2$$

$$= \frac{1}{6}(r+1)\left[r(2r+1) + 6(r+1) \right]$$

$$= \frac{1}{6}(r+1)\left[2r^2 + 7r + 6 \right]$$

$$= \frac{1}{6}(r+1)(r+2)(2r+3) \checkmark$$

Hence, $P(r+1)$.

$\Rightarrow$ $P(n)$, for all n, by math. induction.

d) Let $P(n)$: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$

Base case

$P(1)$: $\frac{1}{1\cdot 2} = \frac{1}{1+1}$ ✓

Inductive step:

Assume $P(k)$: $\frac{1}{1\cdot 2} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

WTS: $P(k+1)$: $\frac{1}{1\cdot 2} + \dots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$.

Now: $\underbrace{\frac{1}{1\cdot 2} + \dots + \frac{1}{k(k+1)}} + \frac{1}{(k+1)(k+2)}$

$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$, by inductive hypothesis.

$= \frac{1}{k+1}\left[k + \frac{1}{k+2}\right]$

$= \frac{1}{(k+1)}\left[\frac{k(k+2)+1}{k+2}\right]$

$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$ ✓

Hence, $P(k+1)$

$\Rightarrow P(n)$, for all n, by math. induction.

e) Let $P(n): \frac{1}{2^n-1} < \frac{3}{2^n}$

<u>Base case:</u>

$P(1): \frac{1}{2^1-1} \overset{?}{=} \frac{1}{1} < \frac{3}{2}$ ✓

<u>Inductive step:</u>

Assume $P(k): \frac{1}{2^k-1} < \frac{3}{2^k}$.

Want to show:

$P(k+1): \frac{1}{2^{k+1}-1} < \frac{3}{2^{k+1}}$

Now: $$\frac{1}{2^{k+1}-1} = \frac{1}{2.(2^k)-1}$$

$$< \frac{1}{2.2^k-2}$$

$$= \frac{1}{2(2^k-1)}$$

$$= \frac{1}{2} \cdot \frac{1}{2^k-1}$$

$$< \frac{1}{2} \cdot \frac{3}{2^k}, \quad \text{by inductive hyp.}$$

$$= \frac{3}{2^{k+1}}$$

Hence, $P(k+1) \Rightarrow P(n)$, for all n, by mathematical induction.

2) TEST:

$2.1+3 = 5 \not< 2^1$

$2.2+3 = 7 \not< 2^2$

$2.3+3 = 9 \not< 2^3$

$2.4+3 = 11 < 2^4$

$2.5+3 = 13 < 2^5$ ✓

Let $P(n)$: $2n+3 < 2^n$

Base case ($n=4$)

$P(4)$: $2.4+3 = 11 < 2^4 = 16$ ✓

Inductive step:

Assume $P(k)$: $2k+3 < 2^k$, some $k \geq 4$

Want to show

$P(k+1)$: $2(k+1)+3 < 2^{k+1}$

Now,

$$
\begin{aligned}
& 2(k+1)+3 \\
&= 2k+5 \\
&= 2k+3+2 < 2^k + 2^1 &&\text{, by inductive hypothesis} \\
&< 2^k + 2^k &&\text{, since } k \geq 4 > 1 \\
&= 2.2^k \\
&= 2^{k+1}
\end{aligned}
$$

Hence, $P(k+1) \Rightarrow P(n)$. for all $n \geq 4$, by mathematical induction.

3a) Let $P(n)$: $a_n < 3$

<u>Base case</u>:

$P(1)$: $a_1 = 2 < 3$ ✓

<u>Inductive Step</u>:

Assume $P(k)$: $a_k < 3$

<u>WTS</u>: $P(k+1)$: $a_{k+1} < 3$

Now, $a_{k+1} = \sqrt{6+a_k}$

$< \sqrt{6+3}$ by ind. hyp.

$= 3$

Hence, $P(k+1) \Rightarrow P(n)$, for all n, by mathematical induction.

b) Let $P(n)$: $a_n < a_{n+1}$

<u>Base case</u>:

$P(1)$: $a_1 = 2 < \sqrt{8} = a_2$

<u>Inductive step</u>:

Assume $P(k)$: $a_k < a_{k+1}$

<u>WTS</u>: $P(k+1)$: $a_{k+1} < a_{k+2}$.

~~Hence~~, ~~[illegible]~~ Now, $a_{k+1} = \sqrt{6+a_k}$

$< \sqrt{6+a_{k+1}} = a_{k+2}$

Hence, $P(k+1) \Rightarrow P(n)$, for all n, by math. induction

c) The sequence is increasing (by (b)) and bounded above (by (a))

$\Rightarrow (a_n)$ convergent by MBT.

Let $L = \lim_{n\to\infty} a_n$.

Then $L = \lim_{n\to\infty} a_{n+1}$

ie
$$\begin{aligned} L &= \lim_{n\to\infty} a_{n+1} \\ &= \lim_{n\to\infty} \sqrt{6+a_n} \\ &= \sqrt{6+\lim a_n}, \text{ by L.L.} \\ &= \sqrt{6+L} \end{aligned}$$

$\Rightarrow L^2 = 6+L$

$\Rightarrow L^2 - L - 6 = 0$

$\Rightarrow (L-3)(L+2) = 0 \quad \Rightarrow L = -2$ or $L = 3$

Since (a_n) increasing, ie

$a_n > a_1 = 2$

$\Rightarrow L = \lim a_n > 2$

$\Rightarrow L = 3$.

4) Let $P(n)$: sum of interior angles of convex n-gon $= (n-2)\pi$

Base case:

$P(1)$: 3-gon = triangle

and sum of ~~are~~ interior angles $= \pi$, by middle school.

Inductive Step:

Assume $P(k)$:

sum of interior angles of convex k-gon $= (k-2)\pi$

WTS: $P(k+1)$:

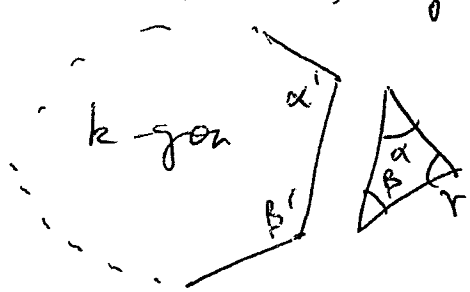
sum of interior angles of convex $(k+1)$-gon

$= (k-1)\pi$.

Now, ~~let~~ consider a $(k+1)$-gon.

Since a convex $(k+1)$-gon can be cut into k-gon and triangle

Then, sum of interior angles of k-gon is $(k-2)\pi$, by inductive hypothesis.



Label interior angles of $(k+1)$-gon by $\alpha_1, \alpha_2, \ldots, \alpha_{k+1}$

Say $\gamma = \alpha_{k+1}$ (see above picture)

and let

$$\alpha_k = \alpha + \alpha'$$
$$\alpha_{k-1} = \beta' + \beta$$

Then;

$$
\begin{aligned}
& \alpha_1 + \alpha_2 + \cdots + \alpha_k + \alpha_{k+1} \\
&= \alpha_1 + \alpha_2 + \cdots + (\beta + \beta') + (\alpha + \alpha') + \alpha_{k+1} \\
&= (\alpha_1 + \alpha_2 + \cdots + \beta' + \alpha') + (\beta + \alpha + \gamma) \\
&= (k-2)\pi + \pi \\
&= (k-1)\pi .
\end{aligned}
$$

Hence, $P(k+1)$.

$\Rightarrow$ $P(n)$, for all $n \geq 3$, by math. induction.

5) It is not true that

$$P(1) \Rightarrow P(2)$$