Middlebury College

Calculus II: Spring 2018

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Supplementary References:
- *Calculus*, Stewart

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An exp-traordinary function

In today's lecture we will define a very interesting function using series. Investigating this function will lead us to the notion of an *inverse function.*

Defining a function via a series:
Let x be any real number and consider the series

$$1+\sum_{n=1}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$

Mathematical workout - flex those muscles
Use the ratio test to show that the above series is (absolutely) convergent, for every real number x.

$$a_n=\frac{x^n}{n!}, \quad \left|\frac{a_{n+1}}{a_n}\right|=\frac{|x|^{n+1}}{(n+1)!}\cdot\frac{n!}{|x|^n}=\frac{|x|}{n+1}\xrightarrow{n\to\infty}0$$

for any x.

By assigning to every real number x the limit of the series $1+\sum_{n=1}^{\infty}\frac{x^n}{n!}$, we have definition for a function

$$(\text{INPUT}) \qquad x \mapsto \exp(x)\overset{def}{=}1+\sum_{n=1}^{\infty}\frac{x^n}{n!} \qquad (\text{OUTPUT})$$

We will call the function $\exp(x)$, defined for every real number x, **the exponential function.**
Remark:

1. Observe that

$$\exp(1)=\sum_{n=0}^{\infty}\frac{1}{n!}=1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\dots$$

This series is a series with positive terms, which implies that its sequence of partial sums (s_m) is strictly increasing. In particular, for any $m = 0, 1, 2, \dots$,

$$s_m < \exp(1) \quad \text{and} \quad \lim_{m\to\infty} s_m = \exp(1).$$

Notice that $s_2 = 1 + 1 + \frac{1}{2} = \frac{5}{2}$ and

$$\sum_{n=2}^{\infty} \frac{1}{n!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots < \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = 1$$

Hence,

$$2.5 = \frac{5}{2} = s_3 < \exp(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots < 1 + 1 + 1 = 3$$

so that

$$2.5 < \exp(1) < 3.$$

In fact, you've seen this number before

$$\exp(1) = e$$

This number is called **Euler's number**, after Leonhard Euler, 1707-1783, a Swiss mathematician and one of the greatest mathematical minds in history.

2. It's possible to show that

$$\exp(1) = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

and, more generally,

$$\exp(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = (\exp(1))^x = e^x$$

In particular,

$$\boxed{\textbf{the exponential function } \exp(x) \textbf{ is } e^x}$$

In fact, the series definition of the function $f(x) = e^x$ was the original definition given by Euler.

Let's investigate some of the basic properties of $\exp(x)$.

CHECK YOUR UNDERSTANDING

Using the definition of $\exp(x)$, show that

1. $\exp(0) = 1$,

$$\exp(0) = 1 + \cancel{\sum_{n=1}^{\infty} \frac{0^n}{n!}}$$

2. $\exp(x) > 1$, for any $x > 0$,

$$\exp(x) = 1 + \underbrace{\left[\sum_{n=1}^{\infty} \frac{x^n}{n!}\right]}_{>0} > 1 + 0 = 1$$

3. $\exp(x) > 1 + x$, for any $x > 0$.

$$\exp(x) = 1 + x + \overbrace{\left[\sum_{n=2}^{\infty} \frac{x^n}{n!}\right]}^{>0} > 1 + x$$

A remarkable property

We are going to investigate a remarkable property of the exponential function. Let x be a real number. For each $m = 1, 2, \ldots$, denote the m^{th} partial sum of the series $\exp(x)$ by $s_m(x)$, so

$$s_m(x) = 1 + \sum_{n=1}^{m} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

We define $s_0(x) = 1$.

CHECK YOUR UNDERSTANDING

Let x, y be real numbers.

1. Write down the expressions for $s_1(x)$, $s_2(x)$.

$$s_1(x) = 1 + x$$
$$s_2(x) = 1 + x + \frac{x^2}{2}$$

2. Show that $s_1(x)s_1(y) = s_1(x+y)$ + *additional terms.*

$$s_1(x)s_1(y) = (1+x)(1+y)$$
$$= 1 + (x+y) + xy = s_1(x+y) + xy$$

3. Show that $s_2(x)s_2(y) = s_2(x+y)$ + *additional terms.*

$$s_2(x)s_2(y) = \left(1 + x + \frac{x^2}{2}\right)\left(1 + y + \frac{y^2}{2}\right) = \underbrace{1 + (x+y) + \frac{\overbrace{(x^2+2xy+y^2)}^{(x+y)^2}}{2}}_{s_2(x+y)} + \frac{x^2y}{2} + \frac{xy^2}{2} + \frac{x^2y^2}{4}$$

4. **Guess the pattern!** Complete the following statement

$$s_3(x)s_3(y) = \underline{s_3(x+y)} + \textit{additional terms}$$

5. **Guess the general pattern!** Complete the following statement: for every $m = 1, 2, \ldots$

$$s_m(x)s_m(y) = \underline{s_m(x+y)} + \textit{additional terms}$$

6. How might you describe the *additional terms* that appeared in your investigations above?

products of form $\frac{x^k}{k!} \cdot \frac{y^l}{l!}$, $k+l > m$.

Recall that

$$\exp(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} = \lim_{m\to\infty} s_m(x).$$

Therefore,

Remarkable Property

$$\exp(x) \cdot \exp(y) = \underline{\exp(x+y)} \qquad (*)$$

Property $(*)$ has lots of remarkable consequences. For example, suppose that x is any positive real number. Then,

$$\begin{aligned} 1 &= \exp(0) \\ &= \exp(x + (-x)) \\ &= \exp(x) \cdot \exp(-x) \end{aligned}$$

In particular,

- $\exp(-x) = \frac{1}{\exp(x)}$, for any real number x.
- $\exp(x) \neq 0$, for any real number x.

CHECK YOUR UNDERSTANDING

1. Use $1 = \exp(x)\exp(-x)$ and the fact that $\exp(x) > 1$, whenever $x > 0$, to deduce that $\exp(x) > 0$, for all x.

2. Let $x < y$ and write $y = x + h$, where $h > 0$. Use $(*)$ to show that $\exp(y) > \exp(x)$. (*Hint: recall that* $\exp(h) > 1$ *whenever* $h > 0$)

   Hence, the exponential function is **strictly increasing.**

3. Based on your investigations, draw the graph of the function $\exp(x)$.