Calculus II: Spring 2018
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Supplementary References:
- *Calculus*, Stewart

Keywords: the exponential function

An exp-traordinary function

In today's lecture we will define a very interesting function using series. Investigating this function will lead us to the notion of an *inverse function*.

Defining a function via a series:

Let x be any real number and consider the series

$$1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Mathematical workout - flex those muscles

Use the ratio test to show that the above series is (absolutely) convergent, for every real number x.

By assigning to every real number x the limit of the series $1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$, we have definition for a function

$$\text{(INPUT)} \qquad x \quad \mapsto \quad \exp(x) \overset{def}{=} 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \qquad \text{(OUTPUT)}$$

We will call the function $\exp(x)$, defined for every real number x, **the exponential function**.

Remark:

1. Observe that

$$\exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots$$

This series is a series with positive terms, which implies that its sequence of partial sums (s_m) is strictly increasing. In particular, for any $m = 0, 1, 2, \ldots$,

$$s_m < \exp(1) \quad \text{and} \quad \lim_{m\to\infty} s_m = \exp(1).$$

Notice that $s_2 = 1 + 1 + \frac{1}{2} = \frac{5}{2}$ and

$$\sum_{n=2}^{\infty} \frac{1}{n!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots < \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots = \frac{1}{2} . \frac{1}{1 - \frac{1}{2}} = 1$$

Hence,

$$2.5 = \frac{5}{2} = s_3 < \exp(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots < 1 + 1 + 1 = 3$$

so that

$$2.5 < \exp(1) < 3.$$

In fact, you've seen this number before

$$\exp(1) = e$$

This number is called **Euler's number**, after Leonhard Euler, 1707-1783, a Swiss mathematician and one of the greatest mathematical minds in history.

2. It's possible to show that

$$\exp(1) = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

and, more generally,

$$\exp(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = (\exp(1))^x = e^x$$

In particular,

the exponential function $\exp(x)$ **is** e^x

In fact, the series definition of the function $f(x) = e^x$ was the original definition given by Euler.

Let's investigate some of the basic properties of $\exp(x)$.

Check your understanding

Using the definition of $\exp(x)$, show that

1. $\exp(0) = 1$,

2. $\exp(x) > 1$, for any $x > 0$,

3. $\exp(x) > 1 + x$, for any $x > 0$.

A remarkable property

We are going to investigate a **remarkable property** of the exponential function. Let x be a real number. For each $m = 1, 2, \ldots$, denote the m^{th} partial sum of the series $\exp(x)$ by $s_m(x)$, so

$$s_m(x) = 1 + \sum_{n=1}^{m} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

We define $s_0(x) = 1$.

Check your understanding

Let x, y be real numbers.

1. Write down the expressions for $s_1(x)$, $s_2(x)$.

2. Show that $s_1(x)s_1(y) = s_1(x + y) +$ *additional terms*.

3. Show that $s_2(x)s_2(y) = s_2(x + y) +$ *additional terms*.

4. **Guess the pattern!** Complete the following statement

$$s_3(x)s_3(y) = \underline{\hspace{4cm}} + \textit{ additional terms}$$

5. **Guess the general pattern!** Complete the following statement: for every $m = 1, 2, \ldots$

$$s_m(x)s_m(y) = \underline{\hspace{4cm}} + \textit{additional terms}$$

6. How might you describe the *additional terms* that appeared in your investigations above?

Recall that

$$\exp(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} = \lim_{m\to\infty} s_m(x).$$

Therefore,

Remarkable Property

$\exp(x) \cdot \exp(y) =$ ____________________ $(*)$

Property $(*)$ has lots of remarkable consequences. For example, suppose that x is any positive real number. Then,

$$\begin{aligned} 1 &= \exp(0) \\ &= \exp(x + (-x)) \\ &= \exp(x) \cdot \exp(-x) \end{aligned}$$

In particular,

- $\exp(-x) = \frac{1}{\exp(x)}$, for any real number x.
- $\exp(x) \neq 0$, for any real number x.

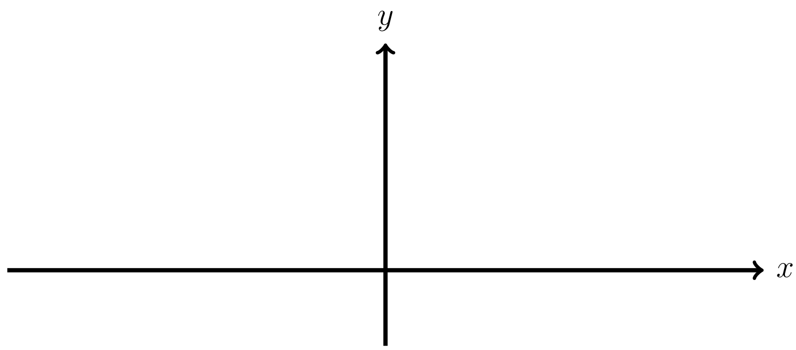
Check your understanding

1. Use $1 = \exp(x)\exp(-x)$ and the fact that $\exp(x) > 1$, whenever $x > 0$, to deduce that $\exp(x) > 0$, for all x.

2. Let $x < y$ and write $y = x + h$, where $h > 0$. Use $(*)$ to show that $\exp(y) > \exp(x)$. (*Hint: recall that* $\exp(h) > 1$ *whenever* $h > 0$)

   Hence, the exponential function is **strictly increasing**.

3. Based on your investigations, draw the graph of the function $\exp(x)$.



Summary

- $\exp(x+y) = \exp(x)\cdot\exp(y)$, for any real numbers x, y.
- $\exp(-x) = \frac{1}{\exp(x)}$, for any real number x.
- $\exp(x) > 0$, for any real number x.
- $\exp(x)$ **is a strictly increasing function**. $(**)$

O Calculus, Where Art Thou?

Let h be a real number and consider the series

$$\frac{\exp(h)-1}{h} = 1 + \frac{h}{2!} + \frac{h^2}{3!} + \ldots = 1 + \sum_{n=1}^{\infty} \frac{h^n}{(n+1)!}.$$

Check your understanding

1. Let $a_n = \frac{h^n}{(n+1)!}$. Use the ratio test to show that the series $1 + \sum_{n=1}^{\infty} a_n$ is (absolutely) convergent.

2. As h gets close, but not equal,to 0, describe what happens to the expression

$$\frac{\exp(h)-1}{h}$$

3. Complete the following statement

$$\lim_{h \to 0} \frac{\exp(h) - 1}{h} = \underline{\hspace{3cm}}$$

Recall what it means for a function $f(x)$ to be *differentiable at* $x = a$: we say that $f(x)$ is **differentiable at** $x = a$ if the following limit exists

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

In this case we write

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

If $f(x)$ is differentiable for every input value x, then we define the **derivative of** $f(x)$ to be the function

$$f'(x) \stackrel{def}{=} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Let a be a real number. Using the **Remarkable Property**, we find

$$\frac{\exp(a+h) - \exp(a)}{h} = \underline{\hspace{9cm}}$$

Hence,

$$\exp'(a) = \lim_{h \to 0} \frac{\exp(a+h) - \exp(a)}{h} = \underline{\hspace{6cm}}$$

Hence,

The function $\exp(x)$ is ____________________ and

$$\frac{d}{dx} \exp(x) = \underline{\hspace{3cm}}$$