Middlebury College

Calculus II: Spring 2018

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π LECTURE

SUPPLEMENTARY REFERENCES:

- *Discrete Mathematics & its Applications*, Rosen: Section 3.2.
- *Series & Induction*, Khan Academy: Induction.

MATHEMATICAL INDUCTION

GET CREATIVE!

We are going to investigate a formula for the sum $\mathcal{O}_n$ of the first n odd natural numbers.

$$\mathcal{O}_n = 1 + 3 + 5 + 7 + 9 + \ldots + (2n - 1)$$

1. Determine $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$.

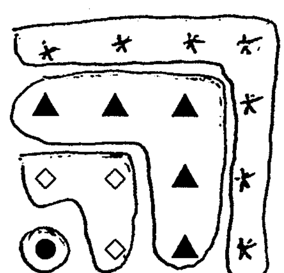
   $\mathcal{O}_1 = 1 = 1^2$ $\quad$ $\mathcal{O}_3 = 9 = 3^2$

   $\mathcal{O}_2 = 4 = 2^2$ $\quad$ $\mathcal{O}_4 = 16 = 4^2$

2. Guess the value of $\mathcal{O}_n$.

   n^2

3. Use the following diagram to explain the value you obtained for $\mathcal{O}_3$.



$1 + 3 + 5 + 7$

4. Using the above diagram, give a similar visual representation of $\mathcal{O}_4$.

5. Using your guess for $\mathcal{O}_n$, explain how you could obtain the value for $\mathcal{O}_{n+1}$. (*You can either explain in words and/or make us of a diagram to justify your explanation*)

Mathematical induction

Many mathematical theorems are of the form

$$P(n), \text{ for every natural number } n. \qquad (*)$$

Here $P(n)$ is some mathematical proposition that depends on n.

Example 0.1. Here are some examples of mathematical propositions $P(n)$ dependent on n.

1. $P(n)$: $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.
2. $P(n)$: $n < 2^n$.
3. $P(n)$: for every real number $r \neq 1$, $r + r^2 + r^3 + \ldots + r^n = \frac{r(1-r^n)}{1-r}$.

CHECK YOUR UNDERSTANDING

Complete the following mathematical proposition:

$P(n)$: *the sum of the first* n odd *natural numbers is* n^2

BRAIN TEASER!

Consider the following situation: Suppose that we have an infinite line of people, called Person 1, Person 2, Person 3, ... etc. We have also been provided with the following information.

B. It is known that Person 1 has received a Snapchat message.

I. It is also known that, whenever Person k receives a Snapchat message they will immediately send a Snapchat message to Person $k+1$.

CHECK YOUR UNDERSTANDING

Provided with the information B and I, we will investigate the following question:

Q: is it possible that there is some person who does not receive a Snapchat?

1. What is your preliminary (i.e. after < 5 seconds thought) answer to the question above: YES or NO?

2. Let's suppose the answer is YES. This means there is some person, Person k say, who does not receive a Snapchat. Assume, further, that Person k is the least such person to not receive a Snapchat i.e. if Person l does not receive a Snapchat then $l \geq k$. Explain why Person $k-1$ must have received a Snapchat message.

   Person k is *first* to not receive message

   ⇒ Person k-1 must receive message

3. Explain why, given all of the information above, Person k must have both received *and* not received a Snapchat.

   By 2, person k has not received message

   By I + (2), Person k receives message

4. What do now believe is the answer to Q?

   No, no person does not receive message

   ⇒ everyone receives message

The proof technique known as **mathematical induction** provides a valid approach to proving statements of the form $(*)$. It proceeds as follows:

1. BASE CASE. Show directly that the proposition $P(1)$ is true.
2. INDUCTIVE STEP. Show that the statement

$$\text{'}if\ P(n)\ then\ P(n+1)\text{'}$$

   holds for any natural number n.

> Determining both the BASE CASE and the INDUCTIVE STEP is *sufficient* to obtain $P(n)$, for every natural number n. The reasoning is similar to the Snapchat message problem above.

Remark: The assumption '*if* $P(n)$' in the INDUCTIVE STEP is called the **inductive hypothesis**.

Example:

Let $P(n)$ be the proposition

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1.$$

We will show, by mathematical induction, that $P(n)$ holds, for every natural number n.

BASE CASE: $P(1)$ is the proposition

$$P(1): \quad \sum_{i=0}^{1} 2^i = 2^2 - 1.$$

This holds since the left hand side is $1 + 2 = 3 = 2^2 - 1$. hence, the BASE CASE is true.

INDUCTIVE STEP: Suppose that $P(k)$ is true, for a natural number k. We want to show that this implies that $P(k+1)$ is true: this will verify the INDUCTIVE STEP.

So, assume that the proposition $P(k)$ is true, for a natural number k. Hence, we know that

$$\sum_{i=0}^{k} 2^i = 2^{k+1} - 1.$$

Consider the proposition

$$P(k+1): \quad \sum_{i=0}^{k+1} 2^i = 2^{(k+1)+1} - 1.$$

We want to show that this proposition holds true. Now, the left hand side of the above expression is

$$\sum_{i=0}^{k+1} 2^i = 2^{k+1} + \sum_{i=0}^{k} 2^i.$$

Since we know, by assumption, that $P(k)$ is true, we can rewrite the right hand side of this expression to get

$$\begin{aligned}\sum_{i=0}^{k+1} 2^i &= 2^{k+1} + \left(2^{k+1} - 1\right) \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{(k+1)+1} - 1.\end{aligned}$$

Hence, if $P(k)$ holds then $P(k+1)$ holds.

Therefore, by mathematical induction, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$, for every natural number n.

CHECK YOUR UNDERSTANDING

Prove the following statements using mathematical induction.

1. The sum of the first n odd integers is n^2, for every natural number n.

$P(n): \quad 1+3+5+7+\dots+(2n-1) = n^2$

Base Case: $P(1): \quad 1 = 1^2$ ✓

Inductive Step: Suppose $P(k)$ is true, for some k.

i.e. $1+3+\dots+(2k-1) = k^2$

Want to show: $P(k+1): \quad 1+3+\dots+(2(k+1)-1) = (k+1)^2$

Now, $\underline{1+3+5+\dots+(2k-1)} + (2(k+1)-1)$

$= k^2 + (2(k+1)-1)$, by inductive hyp.

$= k^2+2k+1 = (k+1)^2$. Hence, $P(k+1)$.

By math. induction, $P(n)$ for all n.

2. For every natural number n, $n < 2^n$.

$P(n): \quad n < 2^n$

Base case: $P(1): \quad 1 < 2^1 = 2$ ✓

Inductive Step: Suppose $P(k)$ holds, for some k.

i.e. $k < 2^k$

Want to show: $P(k+1): \quad (k+1) < 2^{k+1}$

Now

$k+1 < 2^k + 1$, by inductive hyp.

$\leq 2^k + 2^k$, since $2^k \geq 1$

$= 2 \cdot 2^k$

$= 2^{k+1}$. Hence, $P(k+1)$

By mathematical induction, $P(n)$ for all n.