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II HOMEWORK : MATH 122

1a) Let $a_k = k\left(\frac{2}{3}\right)^k$.

- $\left|\frac{a_{k+1}}{a_k}\right| = (k+1)\left(\frac{2}{3}\right)^{k+1} \cdot \left(\frac{3}{2}\right)^k \cdot \frac{1}{k}$

$$= \frac{k+1}{k} \cdot \frac{2}{3} \longrightarrow \frac{2}{3} < 1$$

Hence, convergent by Ratio Test.

- $\sqrt[k]{|a_k|} = \left(k\left(\frac{2}{3}\right)^k\right)^{1/k}$

$$= k^{1/k}\,\frac{2}{3} \longrightarrow \frac{2}{3} < 1, \quad \text{using R.R.}$$

$\Rightarrow$ convergent by Root Test.

b) Let $a_n = \frac{(-3)^n}{2n+1}$

- $\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}}{2n+3} \cdot \frac{2n+1}{3^n} = 3 \cdot \frac{2n+1}{2n+3}$

$$= 3 \cdot \frac{n}{n} \cdot \frac{2 + 1/n}{2 + 3/n}$$

$$\longrightarrow 3 \cdot \frac{2}{2} = 3 > 1$$

Hence, series divergent by Ratio Test

- $\sqrt[n]{|a_n|} = \left(\frac{3^n}{2n+1}\right)^{1/n} = \frac{3}{\sqrt[n]{2n+1}} \longrightarrow \frac{3}{1}$ by R.R.

$\Rightarrow$ divergent by Root Test.

c) Let $a_m = \dfrac{m!}{(-100)^m}$

- $\left|\dfrac{a_{m+1}}{a_m}\right| = \dfrac{(m+1)!}{100^{m+1}} \cdot \dfrac{100^m}{m!}$

$= \dfrac{(m+1)}{100}$, unbounded ~~as~~

$\longrightarrow +\infty$

Hence, series divergent by Ratio Test.

- $\sqrt[m]{|a_m|} = \dfrac{\sqrt[m]{m!}}{100} \longrightarrow +\infty$, by R.R.

unbounded,

Hence, series divergent by Root Test.

d) Let $a_n = \dfrac{10^n}{(n+1)\,4^{2n}} = \dfrac{10^n}{(n+1)\,16^n}$

- $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{10^{n+1}}{(n+2)\,16^{n+1}} \cdot \dfrac{(n+1)16^n}{10^n}$

$= \dfrac{10}{16} \cdot \dfrac{n+1}{n+2} \longrightarrow \dfrac{10}{16} < 1$

Hence, convergent by Ratio Test.

- $\sqrt[n]{|a_n|} = \left(\dfrac{10^n}{(n+1)\,16^n}\right)^{1/n} = \dfrac{10}{16} \cdot \dfrac{1}{\sqrt[n]{n+1}} \longrightarrow \dfrac{10}{16} \cdot \dfrac{1}{1} < 1$

Hence, convergent by Root Test. by R.R.

e) Let $a_n = \dfrac{(n!)^2}{(2n)!}$

- $\left|\dfrac{a_{n+1}}{a_n}\right| = \dfrac{((n+1)!)^2}{(2(n+1))!} \cdot \dfrac{(2n)!}{(n!)^2}$

$= \dfrac{(n+1)^2}{(2n+2)(2n+1)}$

$= \dfrac{\cancel{n^2}(1+1/n)^2}{\cancel{n^2}((2(1+1/n))(2+1/n)}$

$\longrightarrow \dfrac{1+0}{2\cdot(1+0)(2+0)} = \dfrac{1}{4} < 1$

Hence convergent by Ratio Test

- $\sqrt[n]{|a_n|} = \left(\dfrac{(n!)^2}{(2n)!}\right)^{1/n} = \dfrac{(\sqrt[n]{n!})^2}{\sqrt[n]{(2n)!}}$,

don't know how to evaluate limit....

2a) i) Let $a_n = \frac{n}{r^n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{|r|^{n+1}} \cdot \frac{|r|^n}{n}$$

$$= \frac{1}{|r|} \cdot \frac{n+1}{n} \longrightarrow \frac{1}{|r|} < 1$$

Hence, series convergent by Ratio Test.

ii) Since $\sum \frac{n}{r^n}$ convergent,

we know $\lim\limits_{n\to\infty} \frac{n}{r^n} = 0$.

~~2a)~~ iii) $a_n = \frac{n^k}{r^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^k}{|r|^{n+1}} \cdot \frac{|r|^n}{n^k} = \frac{1}{|r|} \cdot \frac{n^k(1+1/n)^k}{n^k}$$

$$\longrightarrow \frac{1}{|r|} < 1$$

Hence, $\sum \frac{n^k}{r^n}$ convergent

$\Rightarrow \lim \frac{n^k}{r^n} = 0$,

b) i) Let $a_n = \frac{r^n}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|r|^{n+1}}{(n+1)!} \cdot \frac{n!}{|r|^n}$$

$$= \frac{|r|}{n+1} \longrightarrow \underset{<1}{0} \quad \text{as } n \to \infty$$

Hence, $\sum \frac{r^n}{n!}$ convergent by Ratio Test.

ii) Since $\sum \frac{r^n}{n!}$ convergent,

$$\lim \frac{r^n}{n!} = 0$$

3) $$\frac{a_{n+1}}{a_n} = \frac{2+\cos(n)}{\sqrt{n}} \geqslant \frac{1}{\sqrt{n}} > 0$$

Since $a_1 = 1 > 0$ then $\frac{a_2}{a_1} > 0 \Rightarrow a_2 > 0$

Similarly $a_3, a_4, a_5, \ldots > 0$

So, $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{a_{n+1}}{a_n} = \frac{2+\cos(n)}{\sqrt{n}}.$$

Now, $$\frac{1}{\sqrt{n}} \leqslant \frac{2+\cos(n)}{\sqrt{n}} \leqslant \frac{3}{\sqrt{n}}$$

Hence, by Squeeze Theorem, $\lim \frac{2+\cos(n)}{\sqrt{n}} = 0$

$\Rightarrow \lim \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$

$\Rightarrow \sum a_n$ converges.

4) a) F - inconclusive

b) T

c) F - if $\sum b_n$ convergent then so is $\sum (-1)^n b_n$, by ACT

d) T - this is the contrapositive statement of ACT.