Middlebury College

Calculus II: Spring 2018

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Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.6.
- *Calculus*, Spivak, 3rd Ed.: Section 23.
- *AP Calculus BC*, Khan Academy: Ratio & alternating series tests.

Keywords: Ratio Test, Root Test.

Series convergence tests V

Today we will conclude our investigation of convergence tests for series.

> **Ratio Test**
>
> Let $\sum a_n$ be a series, where $a_n \neq 0$.
>
> (i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum a_n$ is (absolutely) convergent.
>
> (ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = +\infty$, then the series $\sum a_n$ is divergent.
>
> (iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = 1$, then the test is *inconclusive*: we have gained no additional information on the divergence/convergence of $\sum a_n$.

Example:

1. Consider the series $\sum_{n=1}^{\infty}\frac{n^{10}}{3^n}$. Here $a_n = \frac{n^{10}}{3^n}$, and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{10}}{3^{n+1}}\cdot\frac{3^n}{n^{10}} = \frac{1}{3}\frac{\cancel{n^{10}}(1+1/n)^{10}}{\cancel{n^{10}}} = \frac{1}{3}(1+1/n)^{10}$$

As $n\to\infty$ this last expression tends towards $\frac{1}{3}(1+0)=\frac{1}{3}$. Hence, by the Ratio Test the series converges.

2. Consider the series $\sum_{n=1}^{\infty}\frac{1}{n!}$. Here $a_n = \frac{1}{n!}$, and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(n+1)!}\cdot\frac{n!}{1} = \frac{1}{n+1} \longrightarrow 0 < 1$$

Hence, by the ratio test the series converges.

3. Consider the series $\sum_{n=1}^{\infty} \frac{n!}{n^2(-3)^n}$. Here $a_n = \frac{n!}{n^2(-3)^n}$, and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{(n+1)^2\, 3^{n+1}} \cdot \frac{n^2 \cdot 3^n}{n!} = \frac{1}{3}\,\frac{(n+1)\cdot n^2}{(n+1)^2} = \frac{1}{3}\,\frac{n^2}{n+1} = \frac{1}{3}\left[\frac{n^2-1+1}{n+1}\right]$$

$$= \frac{1}{3}\left[n-1+\frac{1}{n+1}\right] > \frac{1}{3}(n-1) \longrightarrow +\infty$$

Hence, by the ratio test the series diverges

4. Consider the series $\sum_{n=1}^{\infty} \frac{n^3}{\sqrt{n}+2}$. Here $a_n = \frac{n^3}{\sqrt{n}+2}$, and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3(\sqrt{n}+2)}{(\sqrt{n+1}+2)n^3} = \frac{n^{7/2}(1+\frac{1}{n})^3(1+\frac{2}{\sqrt{n}})}{n^{7/2}(\sqrt{1+\frac{1}{n}}+\frac{2}{n})}$$

$$= \frac{(1+\frac{1}{n})^3(1+\frac{2}{\sqrt{n}})}{\sqrt{1+\frac{1}{n}+\frac{2}{n}}} \to \frac{(1+0)^3(1+0)}{\sqrt{1+0}+0} = 1$$

The root test is inconclusive. However, note that we could apply the Limit Comparison Test (for example) and compare with the divergent series $\sum n^{7/2}$ (it's a p-series, with $p = -7/2$) to deduce that the series $\sum a_n$ is divergent.

CHECK YOUR UNDERSTANDING

Use the Ratio Test to determine convergence or divergence of the following series. If the Ratio Test is inconclusive how might you determine convergence?

1. $\sum_{n=1}^{\infty} \frac{n}{8^n}$

$a_n = \frac{n}{8^n}$, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)}{8^{n+1}} \cdot \frac{8^n}{n} = \frac{1}{8}\left(1+\frac{1}{n}\right) \longrightarrow \frac{1}{8} < 1$

Hence, convergent by Ratio test.

2. $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt{n^5+2}}$, $a_n = \frac{n+5}{\sqrt{n^5+2}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+6)}{\sqrt{(n+1)^5+2}} \cdot \frac{\sqrt{n^5+2}}{n+5} = \frac{n+6}{n+5}\sqrt{\frac{n^5+2}{(n+1)^5+2}}$$

$$= \frac{n+6}{n+5}\sqrt{\frac{1+2/n^5}{(1+1/n)^5+2/n^5}}$$

INCONCLUSIVE

$$\longrightarrow 1 \cdot \sqrt{\frac{1+0}{1+0}} = 1$$

3. $\sum_{n=1}^{\infty} \frac{10^n}{n^{10}}$ $\quad a_n = \frac{10^n}{n^{10}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{10^{n+1}}{(n+1)^{10}} \cdot \frac{n^{10}}{10^n} = 10 \cdot \frac{n^{10}}{(n+1)^{10}} = 10 \cdot \frac{1}{(1+\frac{1}{n})^{10}} \longrightarrow 10$$

DIVERGES.

A companion to the Ratio Test is the following:

Root Test

Let $\sum a_n$ be a series, where $a_n \neq 0$.

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum a_n$ is (absolutely) convergent.

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = +\infty$, then the series $\sum a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, then the test is *inconclusive*: we have gained no additional information on the divergence/convergence of $\sum a_n$.

To effectively apply the Root Test we need the following rules:

Root Rules

1. $\lim_{n\to\infty} \sqrt[n]{C} = 1$, for any constant $C > 0$.
2. $\lim_{n\to\infty} \sqrt[n]{n^p} = 1$, for any $p > 0$.
3. $\lim_{n\to\infty} \sqrt[n]{f(n)} = 1$, for any nonzero polynomial $f(n)$ with positive coefficients.
4. $\lim_{n\to\infty} \sqrt[n]{n!} = +\infty$.

Example:

1. Consider the geometric series $\sum_{n=10}^{\infty} \left(\frac{3^n}{5^n}\right)$. Then, $a_n = \frac{3^n}{5^n}$ and

$$\sqrt[n]{\left|\frac{3^n}{5^n}\right|} = \left(\left(\frac{3}{5}\right)^n\right)^{\frac{1}{n}} = \frac{3}{5} \to \frac{3}{5} < 1 \quad \text{as } n \to \infty.$$

   Hence, by the root test, the series is convergent. Of course, we knew this already because this series is a geometric series with $r = \frac{3}{5}$.

2. Consider the series $\sum_{n=1}^{\infty} \frac{(-2)^n n^3}{3^n}$. Then, $a_n = \frac{(-2)^n n^3}{3^n}$ and

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{2^n n^3}{3^n}} = \left(\frac{2^n n^3}{3^n}\right)^{\frac{1}{n}} = \frac{2}{3}\sqrt[n]{n^3} \to \frac{2}{3}\cdot 1 = \frac{2}{3} < 1 \quad \text{as } n \to \infty \text{ by the Root Rules.}$$

   Hence, by the root test the series is convergent.

CHECK YOUR UNDERSTANDING

Use the Root Test to determine convergence of the series above.

1. $\sum_{n=1}^{\infty} \frac{n}{8^n}$

$$\sqrt[n]{|a_n|} = \left(\frac{n}{8^n}\right)^{1/n} = \frac{n^{1/n}}{8} \longrightarrow \frac{1}{8}$$

, by Root Rule 2

Hence, convergent.

2. $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt{n^5+2}}$

$$\sqrt[n]{|a_n|} = \left(\frac{n+5}{\sqrt{n^5+2}}\right)^{1/n} = \frac{(n+5)^{1/n}}{\sqrt{(n^5+2)^{1/n}}} \longrightarrow \frac{1}{\sqrt{1}} = 1$$

by Root Rule 3

inconclusive

3. $\sum_{n=1}^{\infty} \frac{10^n}{n^{10}}$

$$\sqrt[n]{|a_n|} = \left(\frac{10^n}{n^{10}}\right)^{1/n} = \frac{10}{(n^{10})^{1/n}} \longrightarrow \frac{10}{1} = 10$$

by Root Rule 3.

divergent.

Remark: If the Ratio Test can be used to determine convergence of a series $\sum a_n$ (i.e. the test is not inconclusive) then the series will also pass the Root Test (although it may be tricky to find the limit). However, there exists series $\sum a_n$ for which the Ratio Test can't be applied or is inconclusive but the Root Test is conclusive: an example is the series

$$\sum_{n=1}^{\infty} a_n \quad \text{where } a_n = \begin{cases} \frac{1}{2^n}, & n \text{ odd} \\ \frac{4}{2^n}, & n \text{ even} \end{cases}$$

This series is a rearrangement of the (absolutely convergent) geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \dots$$

Then,

$$\frac{a_{n+1}}{a_n} = \begin{cases} 2, & n \text{ odd} \\ 1/8 & n \text{ even} \end{cases}$$

and this sequence does not have a limit. However, $\sqrt[n]{a_n} = 1/2$, for any n, so the Root Test shows this series is convergent.

Appendix

Let $\sum_{n=1}^{\infty} a_n$ be a series, $a_n \neq 0$.

1. The idea behind the Ratio Test is as follows: suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$. Then, as $n \to \infty$, the ratios $\left|\frac{a_{n+1}}{a_n}\right|$ are 'sufficiently close' to L. Now, for any $k < n$ we can write

$$a_n = a_k \cdot \frac{a_{k+1}}{a_k} \cdot \frac{a_{k+2}}{a_{k+1}} \cdot \dots \cdot \frac{a_{n-1}}{a_{n-2}} \cdot \frac{a_n}{a_{n-1}}$$

   and we can find a constant c so that $|a_n|$ can be compared to cL^n, for n sufficiently large. Then, we deduce the behaviour of the series $\sum |a_n|$ by comparing it with the (geometric) series $\sum cL^n$.

2. The idea behind the Root Test is as follows: if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ then, as $n \to \infty$, the terms of the sequence $\left(\sqrt[n]{|a_n|}\right)$ are 'sufficiently close to' L. This means that $|a_n|$ can be compared (in a suitable sense) to L^n. Then, the behaviour of $\sum a_n$ is similar to the behaviour of the geometric series $\sum L^n$.