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March 12 Lecture

Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.6.
- *Calculus*, Spivak, 3rd Ed.: Section 23.
- *AP Calculus BC*, Khan Academy: Ratio & alternating series tests.

Keywords: Ratio Test, Root Test.

Series convergence tests V

Today we will conclude our investigation of convergence tests for series.

Ratio Test

Let $\sum a_n$ be a series, where $a_n \neq 0$.

(i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum a_n$ is (absolutely) convergent.

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = +\infty$, then the series $\sum a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$, then the test is *inconclusive*: we have gained no additional information on the divergence/convergence of $\sum a_n$.

Example:

1. Consider the series $\sum_{n=1}^{\infty} \frac{n^{10}}{3^n}$. Here $a_n = \frac{n^{10}}{3^n}$, and

   $$\left|\frac{a_{n+1}}{a_n}\right| = \rule{6cm}{0.4pt} = \rule{3cm}{0.4pt}$$

   As $n \to \infty$ this last expression tends towards ________________. Hence, by the Ratio Test the series ________________.

2. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n!}$. Here $a_n = \frac{1}{n!}$, and

   $$\left|\frac{a_{n+1}}{a_n}\right| = \rule{6cm}{0.4pt} = \rule{3cm}{0.4pt}$$

   Hence, by the ratio test the series ________________.

3. Consider the series $\sum_{n=1}^{\infty} \frac{n!}{n^2(-3)^n}$. Here $a_n = \frac{n!}{n^2(-3)^n}$, and

$$\left|\frac{a_{n+1}}{a_n}\right| = \rule{8cm}{0.4pt} = \rule{4cm}{0.4pt}$$

Hence, by the ratio test the series ______________

4. Consider the series $\sum_{n=1}^{\infty} \frac{n^3}{\sqrt{n}+2}$. Here $a_n = \frac{n^3}{\sqrt{n}+2}$, and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3(\sqrt{n}+2)}{(\sqrt{n+1}+2)n^3} = \frac{n^{7/2}(1+\frac{1}{n})^3(1+\frac{2}{\sqrt{n}})}{n^{7/2}(\sqrt{1+\frac{1}{n}}+\frac{2}{n})}$$

$$= \frac{(1+\frac{1}{n})^3(1+\frac{2}{\sqrt{n}})}{\sqrt{1+\frac{1}{n}}+\frac{2}{n}} \to \frac{(1+0)^3(1+0)}{\sqrt{1+0}+0} = 1$$

The root test is inconclusive. However, note that we could apply the Limit Comparison Test (for example) and compare with the divergent series $\sum n^{7/2}$ (it's a p-series, with $p = -7/2$) to deduce that the series $\sum a_n$ is divergent.

Check your understanding

Use the Ratio Test to determine convergence or divergence of the following series. If the Ratio Test is inconclusive how might you determine convergence?

1. $\sum_{n=1}^{\infty} \frac{n}{8^n}$

2. $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt{n^5+2}}$

3. $\sum_{n=1}^{\infty} \frac{10^n}{n^{10}}$

A companion to the Ratio Test is the following:

Root Test

Let $\sum a_n$ be a series, where $a_n \neq 0$.

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum a_n$ is (absolutely) convergent.

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = +\infty$, then the series $\sum a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, then the test is *inconclusive*: we have gained no additional information on the divergence/convergence of $\sum a_n$.

To effectively apply the Root Test we need the following rules:

Root Rules

1. $\lim_{n\to\infty} \sqrt[n]{C} = 1$, for any constant $C > 0$.

2. $\lim_{n\to\infty} \sqrt[n]{n^p} = 1$, for any $p > 0$.

3. $\lim_{n\to\infty} \sqrt[n]{f(n)} = 1$, for any nonzero polynomial $f(n)$ with positive coefficients.

4. $\lim_{n\to\infty} \sqrt[n]{n!} = +\infty$.

Example:

1. Consider the geometric series $\sum_{n=10}^{\infty} \left(\frac{3^n}{5^n}\right)$. Then, $a_n = \frac{3^n}{5^n}$ and

$$\sqrt[n]{\left|\frac{3^n}{5^n}\right|} = \left(\left(\frac{3}{5}\right)^n\right)^{\frac{1}{n}} = \frac{3}{5} \to \frac{3}{5} < 1 \quad \text{as } n \to \infty.$$

   Hence, by the root test, the series is convergent. Of course, we knew this already because this series is a geometric series with $r = \frac{3}{5}$.

2. Consider the series $\sum_{n=1}^{\infty} \frac{(-2)^n n^3}{3^n}$. Then, $a_n = \frac{(-2)^n n^3}{3^n}$ and

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{2^n n^3}{3^n}} = \left(\frac{2^n n^3}{3^n}\right)^{\frac{1}{n}} = \frac{2}{3}\sqrt[n]{n^3} \to \frac{2}{3} \cdot 1 = \frac{2}{3} < 1 \quad \text{as } n \to \infty \text{ by the Root Rules.}$$

   Hence, by the root test the series is convergent.

Check your understanding

Use the Root Test to determine convergence of the series above.

1. $\sum_{n=1}^{\infty} \frac{n}{8^n}$

2. $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt{n^5+2}}$

3. $\sum_{n=1}^{\infty} \frac{10^n}{n^{10}}$

Remark: If the Ratio Test can be used to determine convergence of a series $\sum a_n$ (i.e. the test is not inconclusive) then the series will also pass the Root Test (although it may be tricky to find the limit). However, there exists series $\sum a_n$ for which the Ratio Test can't be applied or is inconclusive but the Root Test is conclusive: an example is the series

$$\sum_{n=1}^{\infty} a_n \quad \text{where } a_n = \begin{cases} \frac{1}{2^n}, & n \text{ odd} \\ \frac{4}{2^n}, & n \text{ even} \end{cases}$$

This series is a rearrangement of the (absolutely convergent) geometric series $\sum_{n=1}^{\infty} \frac{1}{2^n}$

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \dots$$

Then,

$$\frac{a_{n+1}}{a_n} = \begin{cases} 2, & n \text{ odd} \\ 1/8 & n \text{ even} \end{cases}$$

and this sequence does not have a limit. However, $\sqrt[n]{a_n} = 1/2$, for any n, so the Root Test shows this series is convergent.

Appendix

Let $\sum_{n=1}^{\infty} a_n$ be a series, $a_n \neq 0$.

1. The idea behind the Ratio Test is as follows: suppose that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$. Then, as $n \to \infty$, the ratios $\left|\frac{a_{n+1}}{a_n}\right|$ are 'sufficiently close' to L. Now, for any $k < n$ we can write

$$a_n = a_k \cdot \frac{a_{k+1}}{a_k} \cdot \frac{a_{k+2}}{a_{k+1}} \cdot \ldots \cdot \frac{a_{n-1}}{a_{n-2}} \cdot \frac{a_n}{a_{n-1}}$$

and we can find a constant c so that $|a_n|$ can be compared to cL^n, for n sufficiently large. Then, we deduce the behaviour of the series $\sum |a_n|$ by comparing it with the (geometric) series $\sum cL^n$.

2. The idea behind the Root Test is as follows: if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ then, as $n \to \infty$, the terms of the sequence $\left(\sqrt[n]{|a_n|}\right)$ are 'sufficiently close to' L. This means that $|a_n|$ can be compared (in a suitable sense) to L^n. Then, the behaviour of $\sum a_n$ is similar to the behaviour of the geometric series $\sum L^n$.