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MATH 122 : HW , MAR. 12

1a) $b_n = \frac{1}{(n!)^2}$; $0 < b_n = \frac{1}{(n!)^2} < \frac{1}{n!} \leq \frac{1}{n}$

By Squeeze Theorem, $\lim b_n = 0$

$$\frac{b_n}{b_{n+1}} = \frac{1}{(n!)^2} \cdot \frac{((n+1)!)^2}{1}$$

$$= \frac{(n+1)!}{n!} \cdot \frac{(n+1)!}{n!} = (n+1)^2 > 1, \text{ for } n = 1, 2, 3, \ldots$$

Hence, $b_n > b_{n+1}$, for $n = 1, 2, 3, \ldots$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{(n!)^2}$ convergent by AST.

b) $b_n = \frac{n^n}{n^{3n}} = \frac{1}{n^{2n}}$

$0 < b_n = \frac{1}{n^{2n}} < \frac{1}{n^n} \leq \frac{1}{n} \Rightarrow \lim b_n = 0$, by Squeeze Theorem.

$$\frac{b_n}{b_{n+1}} = \frac{1}{n^{2n}} \cdot (n+1)^{2(n+1)}$$

$$= \frac{(n+1)^{2n}}{n^{2n}} \cdot (n+1)^2 > (n+1)^2 > 1 \text{ for } n = 1, 2, \ldots$$

$\Rightarrow b_n > b_{n+1}$, for $n = 1, 2, \ldots$

$\Rightarrow \sum (-1)^n \frac{n^n}{n^{3n}}$ convergent by AST.

c) $b_n = \dfrac{8^n}{n^{3n}}$

$$\frac{b_n}{b_{n+1}} = \frac{8^n}{n^{3n}} \cdot \frac{(n+1)^{3(n+1)}}{8^{n+1}}$$

$$= \frac{1}{8} \cdot \frac{(n+1)^{3n}}{n^{3n}} \cdot (n+1)^3 \geq \frac{(n+1)^{3n}}{n^{3n}}, \quad \text{since } (n+1)^3 \geq 8 \text{ if } n \geq 1$$

$$> \left(\frac{n+1}{n}\right)$$

$$> 1$$

$\Rightarrow$ (b_n) decreasing.

Also, for all n, $\dfrac{8}{n} > \dfrac{8}{n^3}$

$$\Rightarrow \quad 0 < \frac{8^n}{n^{3n}} < \frac{8^n}{n^n} = \frac{8 . 8 \ldots 8}{n . n \ldots n}$$

$$< 8^7 \cdot \frac{8}{n}, \quad \text{for all } n$$

By Squeeze Theorem

$$\lim \frac{8^n}{n^{3n}} = 0$$

Hence, series convergent by AST.

d) $0 < b_n = \dfrac{1-n}{3n-n^2} = \dfrac{n-1}{n^2-3n} = \dfrac{n-1}{n(n-3)} < \dfrac{1}{n}, \quad n \geq 4$

$\Rightarrow$ By Squeeze Theorem, $\lim b_n = 0$

Let $f(x) = \dfrac{x-1}{x(x-3)}$, $f'(x) = \dfrac{1}{x(x-3)} - \dfrac{\overbrace{(x-1)(2x-3)}^{2x^2-5x+3}}{x^2(x-3)^2}$

$$= \frac{(x^2-3x) - 2x^2 + 5x - 3}{x^2(x-3)^2}$$

$$= \frac{-x^2+2x-3}{(x(x-3))^2}$$

$$= -\frac{(x-3)(x+1)}{(x(x-3))^2} > 0 \quad \text{for } x \geq 4.$$

Hence, b_n decreasing.

$$\Rightarrow \sum (-1)^n \frac{1-n}{3n-n^2} \quad \text{convergent by AST.}$$

2a) Let $b_n = \frac{1}{(n!)^2}$. Then

$$\frac{b_{n+1}}{b_n} = \frac{1}{((n+1)!)^2} \cdot \frac{(n!)^2}{1}$$

$$= \frac{1}{(n+1)^2} \longrightarrow 0 \quad \text{as } n \to \infty$$

Hence, by Ratio Test, series converges absolutely.

b) Let $a_n = \frac{(-n)^n}{n^{3n}}$.

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{n^n}{n^{3n}}} = \left(\left(\frac{n}{n^3}\right)^n\right)^{1/n}$$

$$= \frac{n}{n^3} = \frac{1}{n^2} \longrightarrow 0$$

Hence, series converges absolutely, by Root Test.

c) $a_n = \left(-\frac{2}{n}\right)^{3n}$

$$\sqrt[n]{|a_n|} = \left(\left(\frac{2}{n}\right)^{3n}\right)^{1/n} = \left(\frac{2}{n}\right)^3 \longrightarrow 0$$

as $n \to \infty$

Hence converges absolutely by Root Test.

d) Consider $\sum_{n=4}^{\infty} \frac{1-n}{3n-n^2}$

$$= \sum_{n=4}^{\infty} \frac{n-1}{n^2-3n}$$

Let $a_n = \frac{n-1}{n^2-3n}$, $b_n = \frac{1}{n}$

Then,

$$\frac{a_n}{b_n} = \frac{(n-1)}{n^2-3n} \cdot \frac{n}{1} = \frac{\cancel{n^2}\left(1-1/n^2\right)}{\cancel{n^2}\left(1-3/n\right)}$$

$$\longrightarrow \frac{1-0}{1-0} = 1 > 0$$

Hence, by LCT, series $\sum \frac{n-1}{n^2-3n}$ divergent

$\Rightarrow$ series conditionally convergent.

3a) b_n is not decreasing (strictly).

$b_1 = 1 \quad b_2 = \frac{1}{4} \quad b_3 = \frac{1}{27}, \quad b_4 = \frac{1}{16}$

b) Let $c_n = \dfrac{1}{(2n+1)^3}, \quad n = 1, 2, 3, \ldots$

$d_n = \dfrac{1}{4^n}$.

Then, $c_n = b_{2n-1}$

$d_n = b_{2n}$

and $\sum_{n=1}^{\infty} (-1)^n b_n = \sum_{n=1}^{\infty} d_n - \sum_{n=1}^{\infty} c_n$

Since $\sum d_n$ convergent (geometric series)

and $\sum c_n$ convergent (LCT with $\sum \frac{1}{n^3}$)

we have $\sum (-1)^n b_n = \sum d_n - \sum c_n$

convergent.

4) Let $b_n = \frac{1}{n^p}$. $\lim \frac{1}{n^p} = 0 \iff p > 0$

Also, $\dfrac{b_n}{b_{n+1}} = \dfrac{1}{n^p} \cdot \dfrac{(n+1)^p}{1} = (1 + 1/n)^p > 1$, whenever $p > 0$

Hence, series convergent when $p > 0$, by AST.

If $p \leq 0$ then $\lim b_n \neq 0 \Rightarrow$ divergent.

Hence, convergent $\Leftrightarrow$ $p > 0$.

5a) T

b) T

c) F, eg $b_n = 1$

d) F.