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February 28 Lecture

Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.2.
- *Calculus*, Spivak, 3rd Ed.: Section 22.
- *AP Calculus BC*, Khan Academy: Partial sums, infinite geometric series.

Keywords: *Harmonic Series, Direct Comparison Test, p-series Test.*

Convergence Tests for Series II

Today we introduce the *Harmonic Series* and begin our investigation into *comparison tests* for series.

Recall the following

> **Test for divergence**
>
> Let $\sum_{n=1}^{\infty} a_n$ be a series. If (a_n) is divergent or (a_n) is convergent and $\lim a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

A common mistake is to assume that the Test for Divergence states the following: *if (a_n) is convergent and $\lim a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ is convergent.* This statement seems like it could be true but the following important example tells otherwise.

The Harmonic Series:

Definition: The series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is called the **Harmonic Series**.

Observe that $\lim_{n\to\infty} \frac{1}{n} = 0$. We investigate the behaviour of the Harmonic Series. Denote the partial sums of the Harmonic Series by H_m, $m = 1, 2, 3, \ldots$.

Spot the pattern!

Consider the following inequalities

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2}, \quad \text{and}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

1. In words, complete the following formulae describing the patterns observed above:

$$\frac{1}{3} + \frac{1}{4} > \frac{\text{no. of}\ \underline{\qquad\qquad\qquad\qquad}}{\text{denom. of}}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{\text{no. of}\ \underline{\qquad\qquad\qquad\qquad}}{\text{denom. of}}$$

2. **Spot the pattern!**

$$\tfrac{1}{9} + \tfrac{1}{10} + \tfrac{1}{11} + \tfrac{1}{12} + \tfrac{1}{13} + \tfrac{1}{14} + \tfrac{1}{15} + \tfrac{1}{16} > \_\_\_\_\_ = \_\_\_\_\_$$

3. **Spot the general pattern!** Complete the following statement: for each $k = 1, 2, 3, \ldots$,

$$\tfrac{1}{2^k+1} + \tfrac{1}{2^k+2} + \tfrac{1}{2^k+3} + \ldots + \tfrac{1}{2^{k+1}-1} + \tfrac{1}{2^{k+1}} > \_\_\_\_\_ = \_\_\_\_\_$$

4. Using 1,2 above explain why

$$H_{2^2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{2} = 1 + \frac{2}{2},$$

$$H_{2^3} = 1 + \frac{1}{2} + \ldots + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2}$$

$$H_{2^4} = 1 + \frac{1}{2} + \ldots + \frac{1}{16} > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{4}{2}$$

5. **Spot the general pattern!** Complete the following statement: for each $k = 1, 2, 3, \ldots$,

$$H_{2^{k+1}} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^{k+1}} > 1 + \_\_\_\_\_$$

6. Complete the following statement

> The sequence of partial sums (H_m) associated to the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ is _______________. Hence, the Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ is _______________.

Remark: You will work through another proof of this result in your Homework.

The Direct Comparison Test (DCT)

In this paragraph we will be concerned with series $\sum a_n$ associated to sequences (a_n) **consisting of positive terms** i.e. for each $n = 1, 2, 3, \ldots$, we require $a_n > 0$.

Check your understanding

Consider the series $\sum_{n=1}^{\infty} \frac{1}{5^n+3}$. Let (s_m) be the sequence of partial terms associated to this series. We are going to investigate the behaviour of this series by comparing it with the known behaviour of the (convergent) geometric series $\sum_{n=1}^{\infty} \frac{1}{5^n}$.

1. Recall that s_m is the sum of the first m terms of the series. For each $m = 1, 2, 3, \ldots$, explain carefully why $s_{m+1} > s_m$. Deduce that (s_m) is an increasing sequence.

2. Consider the geometric series $\sum_{n=1}^{\infty} \frac{1}{5^n}$. Let (t_m) be the associated sequence of partial sums. Explain why (t_m) is convergent and determine the limit L. Explain why $t_m < L$, for $m = 1, 2, 3, \ldots$.

3. Complete the following statement:

> For each $n = 1, 2, 3, \ldots$, we have $5^n < 5^n + 3$ so that $\frac{1}{5^n+3} <$ ________.
>
> Hence, for each $m = 1, 2, 3, \ldots$, $s_m < t_m <$ _____

4. Using what you have discovered in the previous problems, explain carefully why (s_m) is convergent. Deduce that the series $\sum_{n=1}^{\infty} \frac{1}{5^n+3}$ is convergent. Can you determine its limit?

What you have shown in the previous exercise is the idea underlying the proof of the following result.

> **Direct Comparison Test (DCT)**
>
> Let $\sum a_n$ and $\sum b_n$ be series having positive terms.
>
> 1) Suppose that, for each n, $a_n \leq b_n$, and $\sum b_n$ is convergent. Then, $\sum a_n$ is convergent.
>
> 2) Suppose that, for each n, $a_n \geq b_n$, and $\sum b_n$ is divergent. Then, $\sum a_n$ is divergent.

The Direct Comparison Test has the following immediate consequences: in Homework you will show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent; second, we've seen that the Harmonic Series $\sum \frac{1}{n}$ is divergent.

p-series Test

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p is a real number. Then,

1. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p \geq 2$.
2. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent if $p \leq 1$.

Proof:

1. If $p \geq 2$ then, for each $n = 1, 2, 3, \ldots$,

$$n^p \geq n^2 \quad \Longrightarrow \quad 0 < \frac{1}{n^p} \leq \frac{1}{n^2}$$

   Hence, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, we can apply DCT to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is also convergent.

2. If $p \leq 1$ then, for each $n = 1, 2, 3, \ldots$,

$$n^p \leq n \quad \Longrightarrow \quad \frac{1}{n} \leq \frac{1}{n^p}$$

   Hence, since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, we can apply DCT to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is also divergent.

Example:

1. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^3+n+1}$. We are going to compare this series with the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

   For $n = 1, 2, 3, \ldots$, we have

$$n^3 < n^3 + n + 1 \quad \Longrightarrow \quad \frac{1}{n^3 + n + 1} < \frac{1}{n^3}$$

   Since the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent, by the p-series test, the series $\sum_{n=1}^{\infty} \frac{1}{n^3+n+1}$ is convergent, by DCT.

2. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{2n-1} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$$

   We rewrite the summand $\frac{1}{2n-1} = \frac{1}{2}\left(\frac{1}{n-1/2}\right)$. Now, for each $n = 1, 2, 3, \ldots$

$$n - 1/2 < n \quad \Longrightarrow \quad \frac{1}{n} < \frac{1}{n - 1/2}$$

   Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (p-series with $p = 1$), the series $\sum_{n=1}^{\infty} \frac{1}{n-1/2}$ is divergent, by DCT. Hence, the series $\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n-1/2} = \sum_{n=1}^{\infty} \frac{1}{2n-1}$ is divergent.