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MATH 122B: 2/28 HW SOL^{NS}

1a) We can write

$$\frac{1}{n(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$\Rightarrow \quad S_m = \sum_{n=1}^{m} \frac{1}{n(n+2)}$$

$$= \sum_{n=1}^{m} \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$= \frac{1}{2}\sum_{n=1}^{m}\frac{1}{n} - \frac{1}{2}\sum_{n=1}^{m}\frac{1}{n+2}$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} + \dots + \cancel{\frac{1}{m}}\right) - \frac{1}{2}\left(\cancel{\frac{1}{3}} + \cancel{\frac{1}{4}} + \dots + \cancel{\frac{1}{m}} + \frac{1}{m+1} + \frac{1}{m+2}\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{m+1} + \frac{1}{m+2}\right)$$

Then, $$\lim_{m\to\infty} S_m = \lim_{m\to\infty}\left(\frac{3}{4} - \frac{1}{2}\frac{1}{m+1} - \frac{1}{2}\frac{1}{m+2}\right)$$

$$= \frac{3}{4}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty}\frac{1}{n(n+2)} = 3/4$$

b) Similar to above: $\dfrac{1}{n(n+k)} = \dfrac{1}{k}\left(\dfrac{1}{n} - \dfrac{1}{n+k}\right)$

$$S_m = \frac{1}{k}\sum_{n=1}^{m}\frac{1}{n} - \frac{1}{k}\sum_{n=1}^{m}\frac{1}{n+k}$$

$$= \frac{1}{k}\left(1 + \frac{1}{2} + \dots + \frac{1}{m}\right) - \frac{1}{k}\left(\frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{m} + \frac{1}{m+1} + \dots + \frac{1}{m+k}\right)$$

Since we only care what happens as $m \longrightarrow \infty$, we may assume $m > k$

Then,

$$\frac{1}{k}\left(1+\frac{1}{2}+\dots+\frac{1}{k}+\cancel{\frac{1}{k+1}}+\dots+\cancel{\frac{1}{m}}\right)$$

$$-\frac{1}{k}\left(\cancel{\frac{1}{k+1}}+\cancel{\frac{1}{k+2}}+\dots+\cancel{\frac{1}{m}}+\dots+\frac{1}{m+k}\right)$$

$$= \frac{1}{k}\left(1+\frac{1}{2}+\dots+\frac{1}{k}\right)$$

$$-\frac{1}{k}\left(\frac{1}{m+1}+\dots+\frac{1}{m+k}\right)$$

As $m \longrightarrow \infty$, $\frac{1}{m+1}, \frac{1}{m+2}, \dots, \frac{1}{m+k} \longrightarrow 0$

$$\Rightarrow \lim_{m\to\infty} S_m = \frac{1}{k}\left(1+\frac{1}{2}+\dots+\frac{1}{k}\right).$$

ie $$\sum_{n=1}^{\infty} \frac{1}{n(n+k)} = \frac{1}{k}\left(1+\frac{1}{2}+\dots+\frac{1}{k}\right)$$

c) $$\frac{1}{n^2+3n+2} = \frac{1}{(n+1)(n+2)} = \left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$

Then, $$S_m = \sum_{n=1}^{m} \frac{1}{(n+1)(n+2)}$$

$$= \sum_{n=1}^{m} \frac{1}{n+1} - \sum_{n=1}^{m} \frac{1}{n+2}$$

$$= \left(\frac{1}{2}+\cancel{\frac{1}{3}}+\dots+\cancel{\frac{1}{m+1}}\right) - \left(\cancel{\frac{1}{3}}+\cancel{\frac{1}{4}}+\dots+\cancel{\frac{1}{m+1}}+\frac{1}{m+2}\right)$$

$$= \frac{1}{2} - \frac{1}{m+2} \longrightarrow \frac{1}{2}$$

Hence, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} = \frac{1}{2}$.

d) Note: $\displaystyle\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$

$$S_m = \sum_{n=1}^{m} \frac{1}{(2n-1)(2n+1)}$$

$$= \frac{1}{2}\sum_{n=1}^{m} \frac{1}{2n-1} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{2n+1}$$

$$= \frac{1}{2}\left(1 + \cancel{\frac{1}{3}} + \cancel{\frac{1}{5}} + \cancel{\frac{1}{7}} + \dots + \cancel{\frac{1}{2m-1}}\right)$$

$$- \frac{1}{2}\left(\cancel{\frac{1}{3}} + \cancel{\frac{1}{5}} + \cancel{\frac{1}{7}} + \dots + \cancel{\frac{1}{2m-1}} + \frac{1}{2m+1}\right)$$

$$= \frac{1}{2} - \frac{1}{2}\,\frac{1}{2m+1} \longrightarrow \frac{1}{2}$$

Hence, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}$.

2) $\displaystyle\frac{n+1}{n^2+2n} = \frac{n+1}{n(n+2)}$

- Can't write summand as difference of terms of form $\frac{1}{n} - \frac{1}{n+2}$.
- Does not necessarily mean $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2+2n}$ is divergent – just can't use telescoping series argument to determine limit.

3a) $$\lim_{n\to\infty} \frac{n^2}{2n^2+5n+10} = \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}} \cdot \frac{1}{2+5/n+10/n^2}$$

$$= \lim \frac{1}{2+5/n+10/n^2}$$

$$= \frac{1}{2+0+0} = 1/2 \neq 0$$

Hence, divergent by Test for Div.

b) $$\frac{1}{n^2-1} = \frac{1}{(n-1)(n+1)} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

Then, $$\sum_{n=2}^{m} \frac{1}{n^2-1} = \frac{1}{2}\sum_{n=2}^{m} \frac{1}{n-1} - \frac{1}{2}\sum_{n=2}^{m}\frac{1}{n+1}$$

$$= \frac{1}{2}\left(1+\frac{1}{2}+\dots+\cancel{\frac{1}{m-1}}\right) - \frac{1}{2}\left(\cancel{\frac{1}{3}}+\cancel{\frac{1}{4}}+\dots+\frac{1}{m+1}\right)$$

$$= \frac{1}{2}(1+1/2) - \frac{1}{2}\frac{1}{m} - \frac{1}{2}\frac{1}{m+1}$$

$$\longrightarrow \frac{3}{4} - 0 - 0 = 3/4$$

Hence, $$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}.$$

Also, $$\sum_{n=2}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3}\sum_{n=2}^{\infty}\left(\frac{2}{3}\right)^n$$

$$= \frac{1}{3} \cdot \frac{\left(\frac{2}{3}\right)^2}{1-2/3},$$

$$= \frac{4}{9}.$$

Since $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ and $\sum_{n=2}^{\infty} \frac{2^n}{3^{n+1}}$ convergent,

so is $\sum_{n=2}^{\infty} \frac{1}{n^2-1} + \frac{2^2}{3^{n+1}}$ $\left(= \frac{3}{4} + \frac{4}{9} \right)$

c) $\lim_{n \to \infty} \frac{3n^2-1}{n^2} = \lim_{n \to \infty} \left(3 - 1/n^2\right) = 3 \neq 0$

Divergent, by Test for divergence.

d) $\sum_{n=1}^{\infty} \frac{3}{2n(n+2)} = \frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ ← convergent by 1a)

$\sum_{n=1}^{\infty} \frac{5}{3n(n+3)} = \frac{5}{3} \sum_{n=1}^{\infty} \frac{1}{n(n+3)}$ ← convergent by 1b) ($k=3$)

$\Rightarrow \sum_{n=1}^{\infty} \frac{3}{2n(n+2)} + \frac{5}{3n(n+3)}$ convergent.

4) a) T — $\sum_{n=1}^{\infty} 1$

b) F — $\sum \frac{1}{n}$

c) F — $a_n = n$, $b_n = -n$

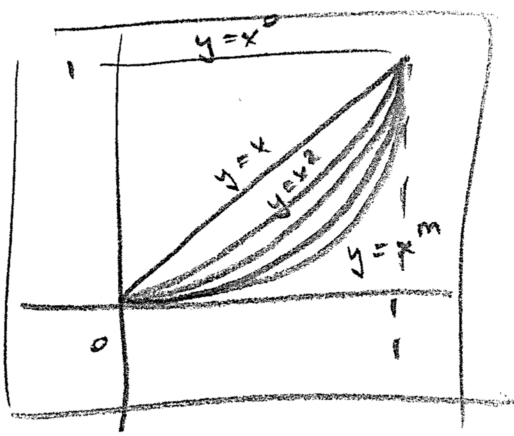
d) F — $a_n = 1$.

5) $$S_m = \sum_{n=1}^{m} \frac{1}{n(n+1)}$$

$$= \sum_{n=1}^{m} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

Then $$\frac{1}{n} - \frac{1}{n+1} = \int_0^1 (x^{n-1} - x^n)\, dx$$

= area between $y = x^{n-1}$ and $y = x^n$, $0 \le x \le 1$



ie S_m = sum of areas between $y = x^{n-1}$ and $y = x^n$, $n = 0, 1, 2, .., m$, $0 \le x \le 1$

As $m \to \infty$, the areas fill out unit square i.e. area = 1

$$\Rightarrow \lim_{m \to \infty} S_m = 1$$