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February 26 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.2.
- *Calculus*, Spivak, 3rd Ed.: Section 22.
- *AP Calculus BC*, Khan Academy: Partial sums, infinite geometric series.

Keywords: *convergence tests for series, telescoping series, test for divergent series.*

Convergence Tests for Series I

Today we investigate how to determine convergence of a type of series known as a **telescoping series**.

Recall: given a sequence (a_n) we associate the **series** $\sum_{n=1}^{\infty} a_n$. The series $\sum_{n=1}^{\infty} a_n$ is **convergent** if its associated sequence of partial sums (s_m), where

$$s_m = a_1 + \ldots + a_m,$$

is convergent. In this case, the convergent series is assigned the limit of (s_m),

$$\sum_{n=1}^{\infty} a_n = \lim_{m\to\infty} s_m$$

Otherwise, the series $\sum_{n=1}^{\infty} a_n$ is **divergent**.

We saw on February 22 how to determine convergence of **geometric series**: these are series of the form $\sum_{n=1}^{\infty} r^n$. Recall that, when $r \neq 1$, we were able to find a nice formula for the m^{th} partial sum s_m of a geometric series

$$s_m = \frac{r(1 - r^{m+1})}{1 - r}$$

We can use the Geometric Progression Theorem to determine convergence of the right hand side: since $\lim_{m\to\infty} r^{m+1} = 0$, whenever $|r| < 1$, we found

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}, \quad \text{whenever } |r| < 1.$$

Geometric Series Theorem

- Let $|r| < 1$. Then, the geometric series $\sum_{n=1}^{\infty} r^n$ is convergent and $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$.
- Let $|r| > 1$. Then, the geometric series $\sum_{n=1}^{\infty} r^n$ is divergent.

In general it can be a tricky task to determine the convergence of a series using the sequence of partial sums: it is usually very difficult to find a nice formula for s_m. However, there are some exceptions which we will now investigate.

Example:

1. Consider the series $\sum_{n=1}^{\infty} 1$. Then, the m^{th} partial sum is

$$s_m = 1 + 1 + \ldots + 1 = m$$

   Since the sequence $(s_m) = (1, 2, 3, 4, \ldots)$ is unbounded, the sequence (s_m) is ____________________. Hence, the series $\sum_{n=1}^{\infty}$ is ____________________.

2. Consider the series $\sum_{n=1}^{\infty}(-1)^n$. The sequence of partial sums (s_m) associated to this series are

$$s_1 = -1, \quad s_2 = -1 + 1 = 0, \quad s_3 = -1 + 1 - 1 = -1, \quad s_4 = -1 + 1 - 1 + 1 = 0, \ldots$$

   In general,

$$s_m = \begin{cases} -1, & \text{if } m \text{ odd,} \\ 0, & \text{if } m \text{ even.} \end{cases}$$

   The sequence (s_m) does not converge so that series $\sum_{n=1}^{\infty}(-1)^n$ is divergent.

Check your understanding

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

1. Determine the first five partial sums s_1, s_2, s_3, s_4, s_5 as a fraction in simplest terms.

2. What do you expect to be the expression for s_m, the m^{th} partial sum?

3. Based on your guess above, is the sequence of partial sums (s_m) convergent or divergent? If convergent, what does this tell us about $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$?; if divergent, give a careful justification.

Remark: The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is an example of a **telescoping series**: the partial sums s_m can be shown to be a difference of two similar sums with successive cancellation.

Example:

1. Consider the series
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$
Note that we can rewrite the summand as
$$\frac{1}{n(n+1)} = \underline{\hspace{4cm}}$$
Then, the associated partial sums are
$$s_m = \sum_{n=1}^{m} \frac{1}{n(n+1)} = \sum_{n=1}^{m} \underline{\hspace{4cm}} = \underline{\hspace{5cm}}$$
Expaning the sigma notation gives
$$s_m = \underline{\hspace{8cm}} = \underline{\hspace{4cm}}$$
Hence, since $\lim_{m\to\infty} s_m = \underline{\hspace{4cm}}$ we obtain
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \underline{\hspace{3cm}}$$

2. This is a slightly more elaborate example. Consider the series
$$\sum_{n=1}^{\infty} \frac{1}{(3n-1)(3n+2)}$$
Note that we can rewrite the summand as
$$\frac{1}{3}\left(\frac{1}{3n-1} - \frac{1}{3n+2}\right)$$
Then, the partial sums are
$$s_m = \sum_{n=1}^{m} \frac{1}{(3n-1)(3n+2)} = \sum_{n=1}^{m} \frac{1}{3}\left(\frac{1}{3n-1} - \frac{1}{3n+2}\right) = \frac{1}{3}\sum_{n=1}^{m} \frac{1}{3n-1} - \frac{1}{3}\sum_{n=1}^{m} \frac{1}{3n+2}$$
Expanding the sigma notation gives
$$s_m = \frac{1}{3}\left(\frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \ldots + \frac{1}{3m-1}\right) - \frac{1}{3}\left(\frac{1}{5} + \frac{1}{8} + \ldots + \frac{1}{3(m-1)+2} + \frac{1}{3m+2}\right)$$
$$= \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3m+2}\right)$$
Hence, since $\lim_{m\to\infty} s_m = \lim_{m\to\infty} \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3m+2}\right) = \frac{1}{6}$, we obtain
$$\sum_{n=1}^{\infty} \frac{1}{(3m-1)(3m+2)} = \frac{1}{6}$$

In the next few lectures we will develop a collection of tests to show that a series is convergent. First, we give a straightforward **test for divergent sequences**.

We make the following observation: let (s_m) be the sequence of partial sums associated to the series $\sum_{n=1}^{\infty} a_n$. Then, we can recover the sequence (a_n) from the sequence of partial sums by noting that

$$\begin{aligned} a_1 &= s_1, \\ a_2 &= (a_1 + a_2) - a_1 = s_2 - s_1, \\ a_3 &= (a_1 + a_2 + a_3) - (a_1 + a_2) = s_3 - s_2, \\ &\vdots \end{aligned}$$

Hence, for each $n = 1, 2, 3, \ldots$,

$$a_{n+1} = (a_1 + a_2 + \ldots + a_{n+1}) - (a_1 + \ldots + a_n) = s_{n+1} - s_n.$$

Mathematical workout - flex those muscles!

1. Let (b_n) be a convergent sequence, $\lim_{n\to\infty} b_n = L$. Define a new sequence
$$(c_n) = (b_2, b_3, b_4, \ldots),$$
so that $c_1 = b_2$, $c_2 = b_3$ etc. Complete the statement:
$$(c_n) \text{ is } \underline{\hspace{4cm}} \text{ and } \lim_{n\to\infty} c_n = \underline{\hspace{3cm}}.$$

2. Let (s_m) be the sequence of partial sums associated to the series $\sum_{n=1}^{\infty} a_n$. Assume that $\sum_{n=1}^{\infty} a_n$ is convergent.
   (a) Using the previous exercise, explain carefully why $\lim_{n\to\infty}(s_{n+1} - s_n) = 0$.

   (b) Complete the following statement:

   > If the series $\sum_{n=1}^{\infty} a_n$ is convergent then (a_n) is convergent and $\lim_{n\to\infty} a_n =$____________.

Considering the *contrapositive statement*[1] we obtain the following

Test for divergence

> Let $\sum_{n=1}^{\infty} a_n$ be a series. If (a_n) is divergent or $\lim a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example: Consider the series

$$\sum_{n=1}^{\infty} \frac{2n^2 + 1}{5n^2 + 6n + 1}.$$

This is the series associated to the sequence (a_n), where $a_n = \frac{2n^2+1}{5n^2+6n+1}$. Since

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^2 + 1}{5n^2 + 6n + 1} = \frac{2}{5} \neq 0,$$

the series $\sum_{n=1}^{\infty} \frac{2n^2+1}{5n^2+6n+1}$ does not converge, by the test for divergence.

[1] Given a statement of the form *if P then Q*, the *contrapositive statement* is the logically equivalent statement *if 'not Q' then 'not P'*. For example, the statement '*if you are a Vermonter then you are American*' is logically equivalent to '*if you are not an American then you are not a Vermonter*'.