HW MATH 122B : FEB. 26

1a) $$a_n - a_{n+1} = \frac{4}{(n+1)^3} - \frac{4}{(n+2)^3}$$

$$= 4\left(\frac{1}{(n+1)^3} - \frac{1}{(n+2)^3}\right) > 0$$

since $\frac{1}{n+1} > \frac{1}{n+2}$.

Hence, $a_n > a_{n+1}$ and (a_n) decreasing

Also, $a_n > 0$, for all n, so (a_n) convergent by MBT

b) $$a_n - a_{n+1} = \frac{n+1}{5^n} - \frac{n+2}{5^{n+1}}$$

$$= \frac{5(n+1) - (n+2)}{5^{n+1}}$$

$$= \frac{4n+3}{5^{n+1}} > 0 \quad , \text{ for } n = 1, 2, 3, \ldots$$

Hence, $a_n > a_{n+1}$ and (a_n) decreasing

Since $a_n > 0$, for $n = 1, 2, \ldots$,

we have (a_n) convergent, by MBT.

c) $$a_n = \frac{n}{n+3} = \frac{n+3-3}{n+3} = 1 - \frac{3}{n+3}$$

$$a_n - a_{n+1} = 1 - \frac{3}{n+3} - \left(1 - \frac{3}{n+4}\right)$$

$$= \frac{3}{n+4} - \frac{3}{n+3}$$

$$= 3\left(\frac{1}{n+4} - \frac{1}{n+3}\right) < 0 \,, \quad \text{since } \frac{1}{n+4} < \frac{1}{n+3} \,,$$

for all n.

Hence, $a_n < a_{n+1}$ is increasing.

Since $a_n = 1 - \frac{4}{n+3} < 1$, for $n = 1, 2, \ldots$

we have (a_n) convergent by MBT.

d) $a_n = \frac{n}{2n+1} = \frac{n + \frac{1}{2} - \frac{1}{2}}{2(n+\frac{1}{2})}$

$$= \frac{1}{2} - \frac{1}{4}\frac{1}{n+\frac{1}{2}}$$

Similarly, (a_n) increasing and bounded above by $\frac{1}{2}$ $\Rightarrow$ convergent by MBT.

2) a) $1! = 1$

$2! = 2$

$3! = 6$

$4! = 24$

$5! = 120$

$6! = 720$

$7! = 5040$

$8! = 40320$

b) $a_1 = 2 \qquad a_2 = 2 \qquad a_3 = \frac{8}{6} = \frac{4}{3} \qquad a_4 = \frac{16}{24} = \frac{2}{3}$

$a_5 = \frac{32}{120} = \frac{4}{15}$

c) $\frac{b_n}{b_{n+1}} \le 1 \iff b_n \le b_{n+1}$ ie (b_n) increasing

$\frac{b_n}{b_{n+1}} \ge 1 \iff b_n \ge b_{n+1}$ ie (b_n) decreasing

d)
$$\frac{a_n}{a_{n+1}} = \frac{2^n}{n!} \cdot \frac{(n+1)!}{2^{n+1}}$$
$$= \frac{2^n}{2^{n+1}} \cdot \frac{(n+1)!}{n!}$$
$$= \frac{1}{2} \cdot \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots n(n+1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdots n}$$
$$= \frac{1}{2}(n+1) \geqslant 1 \qquad \text{for } n = 1, 2, 3, 4, \ldots$$

Hence, (a_n) decreasing.
Since $a_n > 0$ for all n, (a_n)
convergent by MBT.

$a_1 = 2 < \frac{4}{1}$

$a_2 = 2 = \frac{4}{2}$

e)
$$\frac{2^n}{n!} = \frac{2 \cdot \cancel{2} \boxed{2} \cdots \boxed{\cdot\ 2}\, 2}{1 \cdot \cancel{2} \underbrace{\boxed{3}}_{\le 1} \cdots \underbrace{\boxed{(n-1)}}_{\le 1} n}$$

for $n \geqslant 3$
$$< \frac{2 \cdot 2}{n} = \frac{4}{n}$$

Hence, since $0 < a_n \le 4/n$ and

$$\lim_{n\to\infty} 0 = \lim_{n\to\infty} 4/n = 0 ,$$

by Squeeze Theorem, we have

$$\lim a_n = 0 .$$

3) a) $\sum_{n=1}^{\infty} a_n$ convergent if its sequence of partial sums (S_m), where $S_m = a_1 + a_2 + \dots + a_m$, is convergent.

b) Since $S_m = \dfrac{m^2+1}{3m^2+2m-1}$

$$= \frac{\cancel{m^2}}{\cancel{m^2}} \cdot \frac{1 + 1/m^2}{3 + 2/m - 1/m^2}$$

we use L.L. to show

$$\lim S_m = \frac{1}{3}$$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges to $1/3$.

c) Since $\sum_{n=1}^{\infty} r^n = \dfrac{r}{1-r}$ we have

$$\sum_{n=0}^{\infty} r^n = 1 + \sum_{n=1}^{\infty} r^n = 1 + \frac{r}{1-r}$$

$$= \frac{1-r+r}{1-r} = \frac{1}{1-r} .$$

d) $\sum_{n=k}^{\infty} r^n$ is series associated to $(r^k, r^{k+1}, r^{k+2}, \ldots)$

$$= r^{k-1}(r, r^2, r^3, \ldots)$$

Hence,

$$\sum_{n=k}^{m} r^n = \sum_{n=1}^{m-k+1} r^{k-1} \cdot r^n$$

$$= r^{k-1}\left(\sum_{n=1}^{m-k+1} r^n\right)$$

~~Let~~ Since k is fixed, as $m \to \infty$ we have $m-k+1 \to \infty$

$$\Rightarrow \lim_{m\to\infty}\left(\sum_{n=1}^{m-k+1} r^n\right) = \lim_{m\to\infty} \sum_{n=1}^{m} r^n$$

$$= \frac{r}{1-r}.$$

Hence,

$$\lim_{m\to\infty} \sum_{n=k}^{m} r^n = \lim_{m\to\infty} r^{k-1} \sum_{n=1}^{m-k+1} r^n$$

$$= r^{k-1} \cdot \lim_{m\to\infty} \sum_{n=1}^{m} r^n$$

$$= r^{k-1} \cdot \frac{r}{1-r} = \frac{r^k}{1-r}.$$