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February 22 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.1-11.2.
- *Calculus*, Spivak, 3rd Ed.: Section 22.
- *AP Calculus BC*, Khan Academy: Partial sums, infinite geometric series.

Keywords: *geometric progression, series, partial sums, convergent series, divergent series, geometric series.*

Sequences: Geometric Progressions. Introduction to Series

Today we determine convergence of a class of sequences known as **geometric progressions**. We will also take our first steps into the realm of **series**.

First we recall a result from February 21 Homework.

Test for Divergent Sequences:

> Let (a_n) be a sequence. If (a_n) is unbounded then (a_n) is divergent.

Geometric progressions:

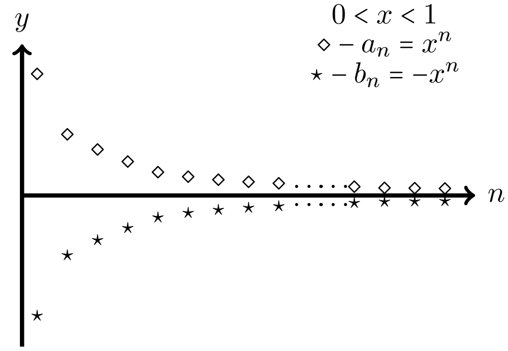
Suppose that $0 \leq x \leq 1$. Consider the sequence (a_n), where $a_n = x^n$. Such a sequence is called a **geometric progression**. We say in February 21 Lecture that (a_n) is convergent and $\lim_{n\to\infty} a_n = 0$. We also showed the following:

1. Let $0 \leq x < 1$, Consider the sequence (b_n), where $b_n = -x^n$. Then (b_n) is monotonic (increasing) and bounded ($-1 \leq b_n \leq 0$), hence convergent by Monotonic Bounded Theorem.

2. Let $x > 1$ and define the sequence (c_n), where $c_n = x^n$. This sequence is increasing and unbounded. Hence, (c_n) is divergent, by the Test for Divergent Sequences.

3. Let $x < -1$ and define the sequence (d_n), where $d_n = x^n$. This sequence is unbounded. Hence, (d_n) is divergent, by the Test for Divergent Sequences.

Now, suppose that $-1 < y < 0$, and let $x = |y|$.



Circle the points on the above graph corresponding to the sequence $(y^n) = (y, y^2, y^3, \dots)$.

Remark 1. In general, a sequence (a_n) is a **geometric progression** if there is a real number x satisfying

$$\frac{a_{n+1}}{a_n} = x, \quad \text{for every } n = 1, 2, 3, \dots.$$

Fact: Every geometric sequence is of the form

$$(cx, cx^2, cx^3, \dots)$$

for some constant c and real number x.

CREATE YOUR OWN THEOREM!

Geometric Progression Theorem (GPT)

> Let x be a real number, c a constant, and consider the geometric progression $(cx^n) = (cx, cx^2, xc^3, \dots)$.
>
> 1. Let $-1 < x < 1$. Then, (cx^n) is ________________________ and $\lim_{n\to\infty} x^n =$ __________.
> 2. Let $|x| > 1$. Then, (cx^n) is ________________________.
> 3. Let $x = 1$. Then, (cx^n) is ________________________ and $\lim_{n\to\infty} x^n =$ __________.
> 4. Let $x = -1$. Then, (cx^n) is ________________________.

Use some of the following phrases/symbols to complete the proof of the first proposition above.

'Squeeze Theorem' **'Monotonic Bounded Theorem'** **'decreasing'** **0**

'convergent' **'divergent'** **'increasing'** **'bounded above'** **1** **∞**

Proof of 1.

Let $-1 < x < 1$ and write $r = |x|$. By the ____________________________,

the sequences (cr^n) and $(-cr^n)$ are ____________________. Moreover,

$$\lim_{n\to\infty} cr^n = \lim_{n\to\infty} -cr^n = \_\_\_\_\_\_\_\_\_\_.$$

Hence, using the ________________________, the sequence (cx^n) is ___________________

and $\lim_{n\to\infty} cx^n =$ __________.

Example 2. 1. Consider the sequence

$$(-\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \frac{16}{81}, \ldots)$$

This is a geometric progression (x^n) with $x = -\frac{2}{3}$. Hence, since $-1 < x < 1$, the sequence is convergent with limit 0, by the Geometric Progression Theorem.

2. Consider the sequence (a_n), where $a_n = \frac{4^n}{3^{n+2}}$. This is a geometric progression since

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}}{3^{n+3}}\frac{3^{n+2}}{4^n} = \frac{4}{3}, \quad \text{for } n = 1, 2, 3\ldots$$

In fact, we observe that

$$a_n = \frac{4^n}{3^{n+2}} = \frac{1}{3^2}\frac{4^n}{3^n} = \frac{1}{9}\left(\frac{4}{3}\right)^n$$

so that $(a_n) = (cx^n)$, with $c =$ _____, $x =$ _____. By the Geometric Progression Theorem, the sequence is divergent.

Introduction to series:

Every real number x has a decimal expansion and this decimal expansion can have finite or infinite length. For example,

$$\frac{1}{3} = 0.3333333\ldots$$

What does the right hand side of this equality mean?

One way to rewrite the decimal expansion is:

$$0.33333\ldots = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots$$

Again, you might, and should, ask: *what does this mean?* In particular, *what does it mean to 'sum' an infinite number of terms*? This obviously(?) does not make any sense when we consider the sum

$$1 + 2 + 3 + 4 + \ldots = \quad ???$$

First, we have the following basic observation: **it is impossible to 'sum' an infinite number of terms** - there is (literally) not enough time to do so. **'Infinite sums'** are nonsensical in mathematics. However, it is possible to ask whether the sequence of finite sums

$$\begin{aligned} s_1 &= \tfrac{3}{10}, \\ s_2 &= \tfrac{3}{10} + \tfrac{3}{10^2}, \\ &\vdots \\ s_m &= \tfrac{3}{10} + \tfrac{3}{10^2} + \ldots + \tfrac{3}{10^m} \end{aligned}$$

converges to a limit L. In *sigma notation*[1] we have

$$s_m = \frac{3}{10} + \frac{3}{10^2} + \ldots + \frac{3}{10^m} = \sum_{n=1}^{m} \frac{3}{10^n}.$$

Definition: Let (a_n) be a sequence.

1. Define **the m^{th} partial sum associated to** (a_n) to be the (finite) sum

$$s_m = a_1 + a_2 + \ldots + a_m = \sum_{n=1}^{m} a_n.$$

2. Define the **sequence of partial sums associated to** (a_n) to be the corresponding sequence (s_m), where s_m is the m^{th} partial sum associated to (a_n).

3. If (s_m) is convergent then we write

$$\sum_{n=1}^{\infty} a_n \overset{def}{=} \lim_{m\to\infty} s_m = \lim_{m\to\infty} \sum_{n=1}^{m} a_n.$$

   We call the symbol $\sum_{n=1}^{\infty} a_n$ a **series**.

4. More generally, we call the symbol $\sum_{n=1}^{\infty} a_n$ a **series**, even when we don't know whether the associated sequence of partial sums is convergent or not. We say that a series is **convergent** if its associated sequence of partial sums is convergent; we say that a series is **divergent** if it is not convergent.

Important Remark

- A series is the *limit* of a sequence of finite sums.
- **A series being convergent *means* its sequence of partial sums is convergent**.

Check your understanding

1. Let (a_n) be a sequence such that, for each $m = 1, 2, 3, \ldots$, the m^{th} partial sum s_m satisfies

$$s_m = a_1 + a_2 + \ldots + a_m = \frac{2m-1}{3m+5}.$$

   Does the series $\sum_{n=1}^{\infty} a_n$ converge? If so, what is its limit? If not, explain carefully why not.

2. Let (a_n) be a sequence such that the sequence of partial sums associated to (a_n), (s_m), satisfies

$$s_m = 10 - \frac{4}{m^2+1}.$$

   Then, $\sum_{n=1}^{\infty} a_n =$ ________________.

[1] See handout for basic properties of sigma notation.

Since a convergent series is, by definition, the limit of a sequence, we can translate some of the Limit Laws for Sequences into corresponding results for series.

Additive properties of series
Let $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ be convergent series, c a constant. Then, 1. $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$, 2. $\sum_{n=1}^{\infty} ca_n = c\left(\sum_{n=1}^{\infty} a_n\right)$

Geometric series

Definition 3. Let r be a real number and consider the associated geometric progression (r^n). The series $\sum_{n=1}^{\infty} r^n$ is called a **geometric series**.

Let $\sum_{n=1}^{\infty} r^n$ be a geometric series. Then, the m^{th} partial sum s_m associated to this series is

$$s_m = \rule{7cm}{0.4pt}$$

Nifty observation:

$$s_m - rs_m = \rule{4.5cm}{0.4pt}$$

$$= \rule{4.5cm}{0.4pt}$$

$$\Longrightarrow \quad s_m(1-r) = \rule{4.5cm}{0.4pt}$$

In particular, when $r \neq 1$,

$$s_m = \rule{4.5cm}{0.4pt}$$

Therefore, the sequence of partial sums (s_m) associated to the geometric series $\sum_{n=1}^{\infty}$ is **convergent** whenever

$$\rule{3.5cm}{0.4pt}$$

and **divergent** whenever

$$\rule{3.5cm}{0.4pt}$$

Create your own Theorem!

Geometric Series Theorem

Let _____ $< r <$ _____. Then, the geometric series $\sum_{n=1}^{\infty} r^n$ is _______________ and $$\sum_{n=1}^{\infty} r^n = \rule{3cm}{0.4pt}.$$

Example 4. 1. Consider the series

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \dots$$

Using the Additive Properties for Series, we have

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = 3 \sum_{n=1}^{\infty} \frac{1}{10^n}.$$

The series on the right hand side is a geometric series with $r = \frac{1}{10}$. Hence, since $|r| < 1$, the Geometric Series Theorem gives

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = 3\left(\frac{1}{10} . \frac{1}{1 - \frac{1}{10}}\right) = \frac{1}{3}$$

2. The series $\sum_{n=1}^{\infty} (-2)^n 3^{2-n}$ is convergent: indeed, this is the series associated to the sequence (a_n), where

$$a_n = (-2)^n 3^{2-n} = (-2)^n 3^2 3^{-n} = 9\left(\frac{-2}{3}\right)^n .$$

Hence,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 9\left(\frac{-2}{3}\right)^n = 9 \sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)^n$$

and we identify this latter series as a geometric series with $r = \frac{-2}{3}$. As $|r| < 1$, the series is convergent with limit

$$9 \sum_{n=1}^{\infty} \left(\frac{-2}{3}\right)^n = 9\left(\frac{-2}{3}\right) \frac{1}{1 + \frac{2}{3}} = -\frac{18}{5}.$$