Middlebury College

Calculus II: Spring 2018

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Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.1.
- *Calculus*, Spivak, 3rd Ed.: Section 22.
- *AP Calculus BC*, Khan Academy: Infinite sequences.

Sequences: The Monotonic Bounded Theorem

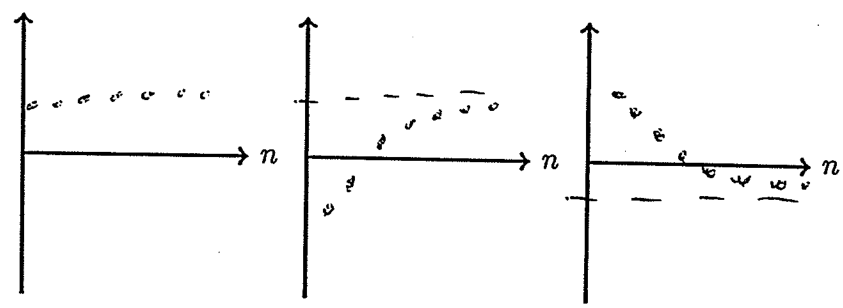
Today we will deduce a useful theorem, the **Monotonic Bounded Theorem** (or **MB Theorem**).

1 Monotonic sequences

Definition 1.1. Let (a_n) be a sequence that is *either* increasing *or* decreasing (or both!). Then, we say that (a_n) is **monotonic.**

Check your understanding

1. Draw the graphs of three (different) monotonic, bounded sequences (a_n), (b_n), (c_n).



2. What common feature do the sequences (a_n), (b_n), (c_n) possess?

Convergent.

Create your own Theorem!

Complete the following statements

Monotonic+Bounded Theorem

Let (a_n) be a monotonic and bounded sequence. Then, (a_n) is CONVERGENT.

More generally,

- if (a_n) is decreasing and BOUNDED BELOW then (a_n) is CONVERGENT.
- if (a_n) is increasing and BOUNDED ABOVE then (a_n) is CONVERGENT.

Example 1.2. 1. Consider the sequence (a_n), where $a_n = \frac{1}{2^n}$. Then, for any natural number n,

$$a_{n+1} = \frac{1}{2^{n+1}} = \frac{1}{2}a_n < a_n.$$

Hence, (a_n) is (strictly) decreasing. Also, $a_n > 0$, for every n, so that (a_n) is bounded below. Hence, by the Monotonic Bounded Theorem the sequence (a_n) is convergent.

2. Consider the sequence (a_n), where $a_n = \cos\left(\frac{\pi(n-1)}{2n}\right)$. Note that

$$0 \le \frac{\pi(n-1)}{2n} = \frac{\pi}{2}\left(1 - \frac{1}{n}\right) < \frac{\pi}{2}\left(1 - \frac{1}{n+1}\right) < \frac{\pi}{2}$$

Hence, since the (differentiable) function $\cos(x)$ is decreasing on the interval $[0,\pi]$, the sequence (a_n) is decreasing. Moreover, a_n is bounded below (by -1, say) so that (a_n) is convergent, by the Monotonic Bounded Theorem.

Remark 1.3. 1. The Monotonic+Bounded Theorem is a little strange: it tells us that a monotonic, bounded sequence is convergent but does not say say how to find $\lim_{n\to\infty} a_n$! Compare this with the Squeeze Theorem where we not only show that a sequence is convergent but also obtain its limit.

2. It can be tricky to check whether a sequence is monotonic, in general. When we are introduced to the technique known as *mathematical induction*, we will have a tool to determine montonicity for a larger class of sequences.

CHECK YOUR UNDERSTANDING

Consider the sequence (a_n), where

$$a_n = \frac{n}{2^n}$$

1. Write down the first five terms of (a_n).

$a_1 = \frac{1}{2}$, $a_2 = \frac{1}{2}$, $a_3 = \frac{3}{8}$, $a_4 = \frac{1}{4}$, $a_5 = \frac{5}{32}$

2. Do you think (a_n) is convergent/divergent? Provide an explanation in support of your claim.

Convergent; denominator will grow faster than numerator.

We will now try to understand the behaviour of the sequence more thoroughly.

3. Show that $2n \geq n+1$, whenever $n \geq 1$.

$n \geq 1 \implies n + n \geq n+1$

$\implies 2n \geq n+1.$

4. Observe that we can write $a_n = \frac{2n}{2^{n+1}}$, for any $n = 1, 2, 3, \ldots$. Using this observation, and the previous problem, show that $a_n \geq a_{n+1}$, for every $n = 1, 2, 3, \ldots$.

$$a_n = \frac{2n}{2^{n+1}} \geq \frac{n+1}{2^{n+1}} = a_{n+1}, \text{ using (3.)}$$

5. Use the Monotonic Bounded Theorem to explain why (a_n) is convergent.

(a_n) is bounded below by 0 and decreasing. Therefore, (a_n) convergent by MBT.

6. Does your argument determine the limit of the convergent sequence (a_n)?

No!

Important Example: Suppose that $0 \leq x \leq 1$. Consider the sequence (a_n), where $a_n = x^n$. Such a sequence is called a **geometric progression**. For each $n = 1, 2, 3, \ldots$

$$a_{n+1} - a_n = x^{n+1} - x^n = x^n(x-1) \leq 0, \quad \text{because } 0 \leq x \leq 1$$

$$\Longrightarrow \quad a_{n+1} \leq a_n, \quad n = 1, 2, 3, \ldots.$$

Hence, (a_n) is <u>decreasing</u>. Also, (a_n) is bounded below: for each $n = 1, 2, 3, \ldots$, we have <u>$a_n \geq 0$</u>. Therefore, by the <u>Monotonic Bounded</u> Theorem the sequence (a_n) is <u>convergent</u>.

In fact, $\lim_{n\to\infty} a_n = 0$ (see Appendix).

CHECK YOUR UNDERSTANDING

1. Let $0 \leq x < 1$, Consider the sequence (b_n), where $b_n = -x^n$. Circle all that apply to (b_n). increasing

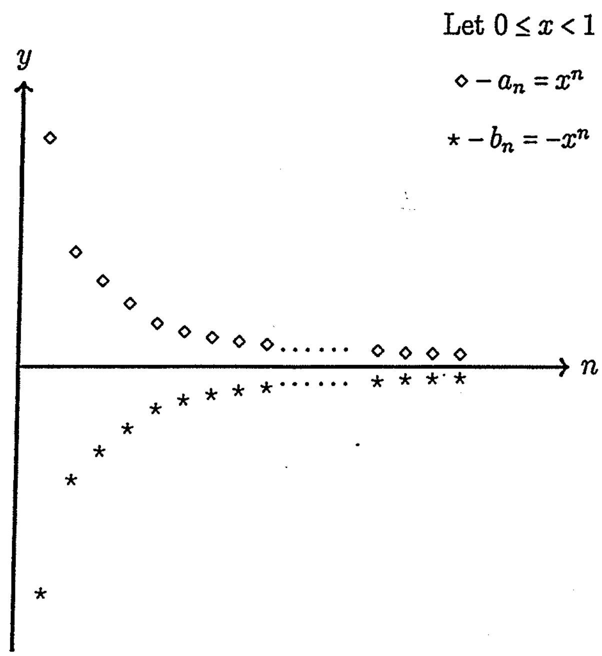
   monotonic   bounded   convergent

2. Let $x > 1$ and define the sequence (c_n), where $c_n = x^n$. Circle all that apply to (c_n).

   monotonic   ~~bounded~~   ~~convergent~~

3. Let $x < -1$ and define the sequence (d_n), where $d_n = x^n$. Circle all that apply to (d_n).

   ~~monotonic~~   ~~bounded~~   ~~convergent~~



Now, suppose that $-1 < x < 0$. **Circle the points on the above graph corresponding to the sequence $(x^n) = (x, x^2, x^3, \ldots)$.**

Remark 1.4. In general, a sequence (a_n) is a **geometric progression** if there is a real number x satisfying

$$\frac{a_{n+1}}{a_n} = x, \quad \text{for every } n = 1, 2, 3, \ldots.$$

Every geometric sequence is of the form

$$(cx, cx^2, cx^3, \ldots)$$

for some constant c and real number x.

Create your own Theorem!

Geometric Progression Theorem (GPT)

> Let x be a real number, c a constant, and consider the geometric progression $(cx^n) = (cx, cx^2, xc^3, \ldots)$.
>
> 1. Let $-1 < x < 1$. Then, (cx^n) is ____________________ and $\lim_{n\to\infty} x^n =$ __________.
>
> 2. Let $|x| > 1$. Then, (cx^n) is ____________________.
>
> 3. Let $x = 1$. Then, (cx^n) is ____________________ and $\lim_{n\to\infty} x^n =$ __________.
>
> 4. Let $x = -1$. Then, (cx^n) is ____________________.

Use some of the following phrases/symbols to complete the proof of the first proposition above.

'Squeeze Theorem' 'Monotonic Bounded Theorem' 'decreasing' 0