Middlebury College

Calculus II: Spring 2018

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Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.1.
- *Calculus*, Spivak, 3rd Ed.: Section 22.
- *AP Calculus BC*, Khan Academy: Infinite sequences.

Sequences: The Monotonic Bounded Theorem

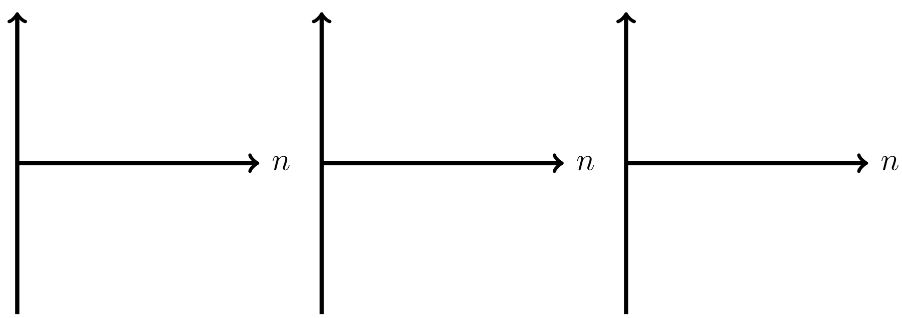
Today we will deduce a useful theorem, the **Monotonic Bounded Theorem** (or **MB Theorem**).

1 Monotonic sequences

Definition 1.1. Let (a_n) be a sequence that is *either* increasing *or* decreasing (or both!). Then, we say that (a_n) is **monotonic**.

Check your understanding

1. Draw the graphs of three (different) monotonic, bounded sequences (a_n), (b_n), (c_n).



2. What common feature do the sequences (a_n), (b_n), (c_n) possess?

Create your own Theorem!

Complete the following statements

Monotonic+Bounded Theorem

Let (a_n) be a monotonic and bounded sequence. Then, (a_n) is ____________________.

More generally,

- if (a_n) is decreasing and ______________________________ then (a_n) is ____________________.
- if (a_n) is increasing and ______________________________ then (a_n) is ____________________.

Example 1.2. 1. Consider the sequence (a_n), where $a_n = \frac{1}{2^n}$. Then, for any natural number n,

$$a_{n+1} = \frac{1}{2^{n+1}} = \frac{1}{2}a_n < a_n.$$

Hence, (a_n) is (strictly) decreasing. Also, $a_n > 0$, for every n, so that (a_n) is bounded below. Hence, by the Monotonic Bounded Theorem the sequence (a_n) is convergent.

2. Consider the sequence (a_n), where $a_n = \cos\left(\frac{\pi(n-1)}{2n}\right)$. Note that

$$0 \leq \frac{\pi(n-1)}{2n} = \frac{\pi}{2}\left(1 - \frac{1}{n}\right) < \frac{\pi}{2}\left(1 - \frac{1}{n+1}\right) < \frac{\pi}{2}$$

Hence, since the (differentiable) function $\cos(x)$ is decreasing on the interval $[0, \pi]$, the sequence (a_n) is decreasing. Moreover, a_n is bounded below (by -1, say) so that (a_n) is convergent, by the Monotonic Bounded Theorem.

Remark 1.3. 1. The Monotonic+Bounded Theorem is a little strange: it tells us that a monotonic, bounded sequence is convergent but does not say say how to find $\lim_{n\to\infty} a_n$! Compare this with the Squeeze Theorem where we not only show that a sequence is convergent but also obtain its limit.

2. It can be tricky to check whether a sequence is monotonic, in general. When we are introduced to the technique known as *mathematical induction*, we will have a tool to determine montonicity for a larger class of sequences.

CHECK YOUR UNDERSTANDING
Consider the sequence (a_n), where

$$a_n = \frac{n}{2^n}$$

1. Write down the first five terms of (a_n).

2. Do you think (a_n) is convergent/divergent? Provide an explanation in support of your claim.

We will now try to understand the behaviour of the sequence more thoroughly.

3. Show that $2n \geq n + 1$, whenever $n \geq 1$.

4. Observe that we can write $a_n = \frac{2n}{2^{n+1}}$, for any $n = 1, 2, 3, \ldots$. Using this observation, and the previous problem, show that $a_n \geq a_{n+1}$, for every $n = 1, 2, 3, \ldots$.

5. Use the Monotonic Bounded Theorem to explain why (a_n) is convergent.

6. Does your argument determine the limit of the convergent sequence (a_n)?

Important Example: Suppose that $0 \leq x \leq 1$. Consider the sequence (a_n), where $a_n = x^n$. Such a sequence is called a **geometric progression**. For each $n = 1, 2, 3, \ldots$

$$a_{n+1} - a_n = x^{n+1} - x^n = x^n(x-1) \leq 0, \quad \text{because } 0 \leq x \leq 1$$

$$\Longrightarrow \quad a_{n+1} \leq a_n, \quad n = 1, 2, 3, \ldots.$$

Hence, (a_n) is ______________. Also, (a_n) is bounded: for each $n = 1, 2, 3, \ldots$, we have ______________. Therefore, by the ________________________ Theorem the sequence (a_n) is ______________.

In fact, $\lim_{n\to\infty} a_n = 0$ (see Appendix).

Check your understanding

1. Let $0 \leq x < 1$, Consider the sequence (b_n), where $b_n = -x^n$. Circle all that apply to (b_n).

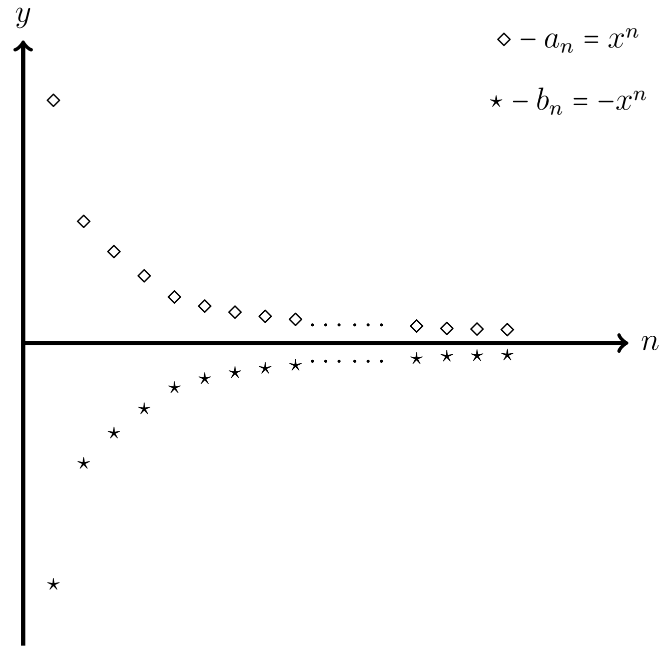
   monotonic **bounded** **convergent**

2. Let $x > 1$ and define the sequence (c_n), where $c_n = x^n$. Circle all that apply to (c_n).

   monotonic **bounded** **convergent**

3. Let $x < -1$ and define the sequence (d_n), where $d_n = x^n$. Circle all that apply to (d_n).

   monotonic **bounded** **convergent**



Now, suppose that $-1 < x < 0$. **Circle the points on the above graph corresponding to the sequence** $(x^n) = (x, x^2, x^3, \ldots)$**.**

Remark 1.4. In general, a sequence (a_n) is a **geometric progression** if there is a real number x satisfying

$$\frac{a_{n+1}}{a_n} = x, \quad \text{for every } n = 1, 2, 3, \ldots.$$

Every geometric sequence is of the form

$$(cx, cx^2, cx^3, \ldots)$$

for some constant c and real number x.

Create your own Theorem!

Geometric Progression Theorem (GPT)

> Let x be a real number, c a constant, and consider the geometric progression $(cx^n) = (cx, cx^2, xc^3, \ldots)$.
>
> 1. Let $-1 < x < 1$. Then, (cx^n) is ________________________ and $\lim_{n\to\infty} x^n =$ __________.
>
> 2. Let $|x| > 1$. Then, (cx^n) is ________________________.
>
> 3. Let $x = 1$. Then, (cx^n) is ________________________ and $\lim_{n\to\infty} x^n =$ __________.
>
> 4. Let $x = -1$. Then, (cx^n) is ________________________.

Use some of the following phrases/symbols to complete the proof of the first proposition above.

'Squeeze Theorem' **'Monotonic Bounded Theorem'** **'decreasing'** **0**

‘**convergent**’ ‘**divergent**’ ‘**increasing**’ ‘**bounded above**’ **1** ∞

Proof of 1.

Let $-1 < x < 1$ and write $r = |x|$. By the ____________________________,

the sequences (cr^n) and $(-cr^n)$ are ___________________. Moreover,

$$\lim_{n\to\infty} cr^n = \lim_{n\to\infty} -cr^n = \_\_\_\_\_\_\_\_\_\_.$$

Hence, using the ________________________, the sequence (cx^n) is ____________________

and $\lim_{n\to\infty} cx^n =$ __________.

Example 1.5. 1. Consider the sequence

$$(-\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \frac{16}{81}, \dots)$$

This is a geometric progression (x^n) with $x = -\frac{2}{3}$. Hence, since $-1 < x < 1$, the sequence is convergent with limit 0, by the Geometric Progression Theorem.

2. Consider the sequence (a_n), where $a_n = \frac{4^n}{3^{n+2}}$. This is a geometric progression since

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}}{3^{n+3}} \frac{3^{n+2}}{4^n} = \frac{4}{3}, \quad \text{for } n = 1, 2, 3 \dots$$

In fact, we observe that

$$a_n = \frac{4^n}{3^{n+2}} = \frac{1}{3^2}\frac{4^n}{3^n} = \frac{1}{9}\left(\frac{4}{3}\right)^n$$

so that $(a_n) = (cx^n)$, with $c = \frac{1}{9}$, $x = \frac{4}{3}$. By the Geometric Progression Theorem, the sequence is divergent.

Appendix: We will show that the convergent sequence (x^n), where $0 \le x < 1$, has limit 0.

Denote the limit of the convergent sequence (x^n) by L: we want to show that $L = 0$. We note the following crucial observation:

the limit of the convergent sequence $(a_n) = (x, x^2, x^3, x^4, \dots)$ is equal to the limit of the convergent sequence $(b_n) = (x^2, x^3, x^4, x^5, \dots)$

We have $b_n = xa_n$, for each $n = 1, 2, 3, \dots$. Thus, using the limit laws, we have

$$L = \lim_{n\to\infty} b_n = \lim_{n\to\infty} (xa_n) = x\left(\lim_{n\to\infty} a_n\right) = xL$$

$$\Longrightarrow \quad L - xL = 0 \quad \Longrightarrow \quad L(x-1) = 0$$

However, we chose $0 \le x < 1$ so that the only way this last equality can hold is if $L = 0$. Hence,

$$\lim_{n\to\infty} x^n = 0, \quad \text{whenever } 0 \le x < 1.$$