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February 19 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.1.
- *Calculus*, Spivak, 3rd Ed.: Section 22.
- *AP Calculus BC*, Khan Academy: Infinite sequences.

Sequences: limits & Theorems

1 Limits Recall what it means for a sequence (a_n) to be convergent with limit L

> for any $\epsilon > 0$, there is some natural number N such that
> $$n \geq N \implies |a_n - L| < \varepsilon. \qquad (*)$$

Remember: in words, the mathematical symbol $|a_n - L|$ means

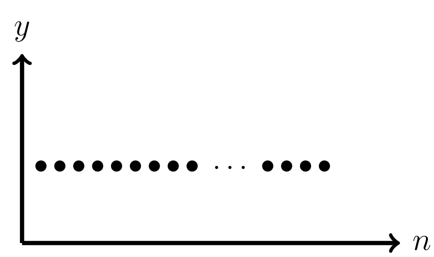
"*the distance from a_n to L*."

You should think about the above statement as saying: no matter how small or large I make my ϵ-band [*for any $\epsilon > 0$*], I can go far enough out to the right in the graph of (a_n) [*there is some natural number N*] so that, beyond that point, the distance between a_n and L is less than ϵ [$n \geq N \implies |a_n - L| < \epsilon$].

Remark 1.1. A basic fact is that, if a sequence (a_n) is convergent with limit L then L is **unique**.

Example 1.2.

1. Let (a_n) be a **constant** sequence - for every n, $a_n = c$, where c is a constant. Then, (a_n) is convergent with limit $L = c$: given any $\epsilon > 0$, we see that
$$n \geq 1 \implies |a_n - L| = |c - c| = 0 < \epsilon.$$
Hence, we can take $N = 1$ for a constant sequence.



2. Consider the sequence (a_n), where $a_n = \frac{1}{n}$. We will show *directly* that (a_n) is convergent with limit $L = 0$.

   Suppose we are given some fixed $\epsilon > 0$. Now, we have to find an N such that, for each $n \geq N$, we necessarily have
$$|a_n - 0| = \left|\frac{1}{n}\right| < \epsilon.$$

Observe that, since $n > 0$, we always have $\left|\frac{1}{n}\right| = \frac{1}{n}$. So, we must find N so that

$$n \geq N \quad \Longrightarrow \quad \frac{1}{n} < \epsilon.$$

Rearranging the above inequality, if we take a natural number $N > \frac{1}{\epsilon}$ then

$$n \geq N \quad \Longrightarrow \quad n \geq N > \frac{1}{\epsilon} \quad \Longrightarrow \quad |a_n| = \frac{1}{n} < \epsilon.$$

Thus, we have shown *directly* that $\lim_{n\to\infty} \frac{1}{n} = 0$ (i.e. that (a_n) satisfies the condition $(*)$).

Corresponding Table: Here's the idea of the above proof: let $\epsilon > 0$ be given. We are using the property $D_{\epsilon,0} : |y| < \epsilon$. Take N large enough so that it satisfies $N > \frac{1}{\epsilon}$. Then,

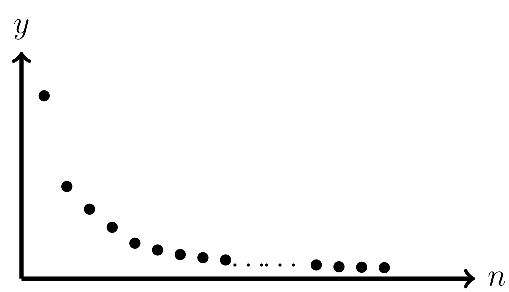
$$\epsilon > \frac{1}{N} > \frac{1}{N+1} > \frac{1}{N+2} > \cdots \text{ etc.}$$

Hence, the corresponding table is

n	1	2	3	$\cdots$	N	$N+1$	$N+2$	$\cdots$
a_n	1	$\frac{1}{2}$	$\frac{1}{3}$	$\cdots$				
T/F	F	F	F	$\cdots$	T	T	T	$\rightarrow$

Check your understanding

Give a graphical formulation of the table above.



2 Theorems Having to prove that a sequence is convergent to a limit L using the above definition is tricky. It can be done for sequences that are defined using basic functions but becomes difficult very quickly. As budding mathematicians, we want to make life easier for ourselves. **This is why Theorems are useful:** *Theorems are power tools that allow us to build truths (i.e. solve problems) in a faster way.*

In this section we will introduce some Theorems that we will use over and over and over and over and... again in this course. These Theorems turn the nasty $\epsilon - N$ approach to determining convergence of a sequence into a far more manageable venture.

(1) Limit Laws for Sequences: Let (a_n), (b_n) be convergent sequences, c a constant. Then, LL1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$,

LL2. $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$,

LL3. $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$,

LL4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, whenever $\lim_{n\to\infty} b_n \neq 0$,

LL5. $\lim_{n\to\infty} a_n^r = (\lim_{n\to\infty} a_n)^r$, if $r > 0$ and $a_n > 0$, for $n = 1, 2, 3, \ldots$.

LL6. Let $f(x)$ be a continuous function whose domain contains $\{a_n\}_{n\geq 1}$ and $\lim_{n\to\infty} a_n$. Then,
$$f(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} f(a_n)$$

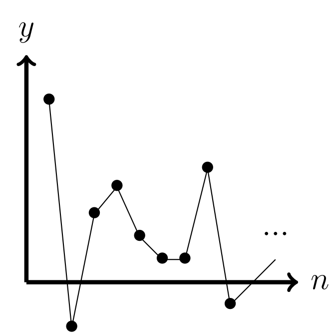
Example 2.1. 1. Let (a_n) be a **constant** sequence - for every n, $a_n = c$, where c is a constant. Then,

2. We've seen above that $\lim_{n\to\infty} \frac{1}{n} = 0$. By LL5 we find, for any $r > 0$,

$$\boxed{\lim_{n\to\infty} \frac{1}{n^r} = \left(\lim_{n\to\infty} \frac{1}{n}\right)^r = 0^r = 0.}$$

3. Consider the sequence (a_n), where $a_n = \frac{2n+1}{n^2+3n}$. We can write
$$a_n = \frac{2n+1}{n^2+3n} = \frac{n}{n^2}.\frac{2+\frac{1}{n}}{1+\frac{3}{n}} = \frac{1}{n}.\frac{2+\frac{1}{n}}{1+\frac{3}{n}}$$
Using the limit laws, and our Example above, we find
$$\lim_{n\to\infty} \frac{1}{n} = \underline{\qquad}, \quad \lim_{n\to\infty}(1+\frac{3}{n}) = \underline{\qquad\qquad}, \quad \lim_{n\to\infty}(2+\frac{1}{n}) = \underline{\qquad\qquad}.$$
Hence,
$$\lim_{n\to\infty} a_n = \left(\lim_{n\to\infty} \frac{1}{n}\right).\left(\frac{\lim_{n\to\infty} 2+\frac{1}{n}}{\lim_{n\to\infty}(1+\frac{3}{n})}\right) = \underline{\qquad\qquad\qquad\qquad}$$

Remark 2.2. We provide a brief outline proving *why* the limit laws hold: let (a_n) be a sequence and consider its graph. Then, it's always possible to find a *continuous function* $f(x)$, defined for all $1 \leq x < \infty$, such that $f(n) = a_n$: the function $f(x)$ whose graph is obtained by drawing straight line segments between (n, a_n), for $n = 1, 2, 3, \ldots$, is such a function.



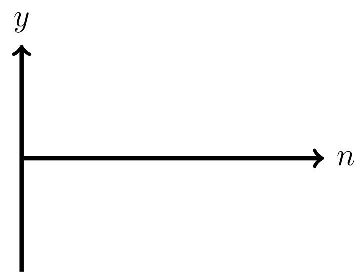
The Limit Laws for Sequences are a consequence of certain Limit Laws for Continuous Functions that you will have seen in your first calculus course.

3 The Squeeze Theorem Consider the sequence (a_n), where

$$a_n = \frac{(-1)^n}{n^2}, \quad n = 1, 2, 3, 4, \dots$$

Check your understanding

1. Draw the graph of (a_n).



2. Do you think the sequence is

convergent **divergent**

Justify your answer.

We will now compare (a_n) to sequences whose behaviour is known. Let $b_n = \frac{1}{n^2}$ and $c_n = -\frac{1}{n^2}$. Then, the sequences (b_n) and (c_n) are convergent and

$$\lim_{n\to\infty} b_n = \underline{\qquad} \qquad \lim_{n\to\infty} c_n = \underline{\qquad}$$

3. Write down a relationship between a_n, b_n, c_n

Create your own Theorem!

The Squeeze Theorem

Let (a_n), (b_n) (c_n) be sequences, where (b_n) and (c_n) are convergent. Let $L = \lim_{n\to\infty} b_n$ and $L' = \lim_{n\to\infty} c_n$ satisfy the relation $\underline{\qquad\qquad}$.

Furthermore, assume that, for each $n = 1, 2, 3, \dots,$, the n^{th} terms a_n, b_n, c_n, satisfy the relation $\underline{\qquad\qquad\qquad\qquad}$.

Then, the sequence (a_n) is $\underline{\qquad\qquad\qquad\qquad}$ and $\lim_{n\to\infty} a_n = \underline{\qquad\qquad}$

Example 3.1. 1. Consider the sequence (a_n), where $a_n = \sin(n)/n$. Define the sequences (b_n), (c_n), where

$$b_n = -\frac{1}{n}, \qquad c_n = \frac{1}{n}, \quad n = 1, 2, 3, \ldots$$

Then, (b_n) and (c_n) are convergent and

$$\lim_{n\to\infty} b_n = 0 = \lim_{n\to\infty} c_n$$

Moreover, for each $n = 1, 2, 3, \ldots$

$$-\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}.$$

Hence, by the Squeeze Theorem, (a_n) is convergent and $\lim_{n\to\infty} a_n = 0$.

2. Consider the sequence (a_n), where $a_n = \frac{1}{2n^2+(-1)^n}$. Observe that, for any natural number n,

$$2n^2 + (-1)^n \geq 2n^2 - 1 \qquad \Longrightarrow \qquad \frac{1}{2n^2 - 1} \geq \frac{1}{2n^2 + (-1)^n}$$

Also, $\frac{1}{2n^2+(-1)^n} > 0$, for any n. If we define the sequences (b_n) and (c_n), with

$$b_n = 0, \quad c_n = \frac{1}{2n^2 - 1}$$

then $b_n < a_n \leq c_n$, for every n. Using the Limit Laws it can be shown (exercise!) that

$$\lim_{n\to\infty} c_n = 0$$

As (b_n) is a constant sequence, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$. Hence, by the Squeeze Theorem, (a_n) is convergent and $\lim_{n\to\infty} = 0$.