Middlebury
College

Calculus II: Spring 2018

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FEBRUARY 16 LECTURE

SUPPLEMENTARY REFERENCES:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.1.
- *Calculus*, Spivak, 3rd Ed.: Section 22.
- *AP Calculus BC*, Khan Academy: Infinite sequences.

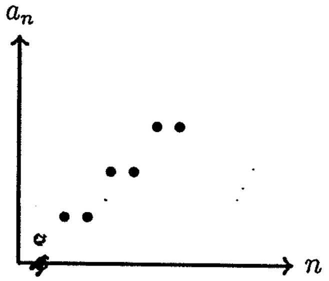
SEQUENCES: LIMITS

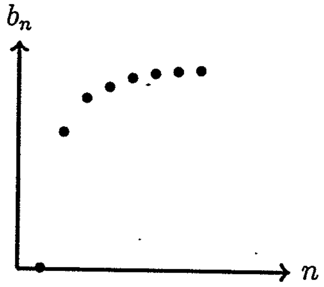
1 Some sequences Recall the sequences from February 15 lecture:

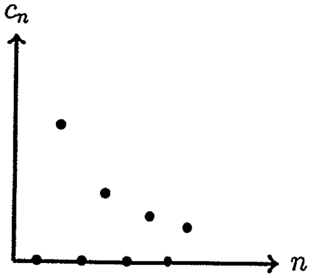
$$(a_n),\quad (b_n),\quad (c_n)$$

where

$$a_n = 2n + (-1)^n,\quad b_n = 1 - \frac{1}{n^2},\quad c_n = 2\left(\frac{1+(-1)^n}{n}\right)$$







CHECK YOUR UNDERSTANDING

For each sequence above, give a property P of real numbers so that P holds for that function as $n \to \infty$.

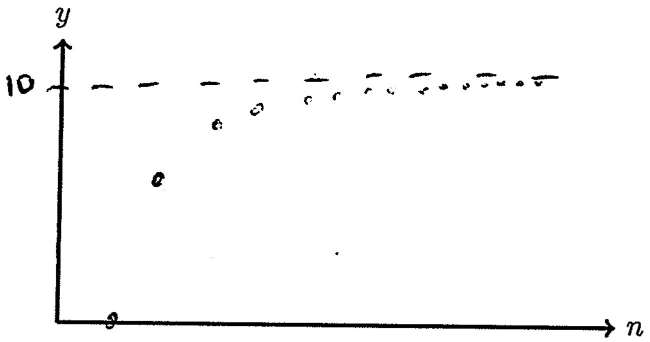
(a_n) P: '$y > 1$'

(b_n) P: '$y < 1$'

(c_n) P: '$|y| < 0.0001$'

2 Limits We introduce the fundamental notion of a **limit** of a sequence. This will build upon our discussion of the phrase 'as $n \to \infty$'.

Consider the increasing and bounded sequence (a_n), where $a_n = 10 - \frac{10}{n}$. Draw the graph of this sequence below and draw the horizontal line $y = 10$.



TRUE/FALSE

T 1. If P is the property $P : 9 < y < 11$ then P holds for (a_n) as $n \to \infty$.

T 2. If P is the property P : *'the distance from y to* 10 *is less than* 0.01*'* then P holds for (a_n) as $n \to \infty$.

T 3. If P is the property $P : 9.9999 < y < 10.0001$ then P holds for (a_n) as $n \to \infty$.

T 4. If P is the property $P : 10 - \frac{1}{2^{100}} < y < 10 + \frac{1}{2^{100}}$ then P holds for (a_n) as $n \to \infty$.

T 5. Let $\varepsilon > 0$ be any positive real number. If P is the property

$$P : \textit{'the distance from } y \textit{ to } 10 \textit{ is less than } \varepsilon\textit{'}$$

then P holds for (a_n) as $n \to \infty$.

We will now formalise the situation observed above.

Let $\varepsilon > 0$ be a positive real number and let L be an arbitrary real number. We define $D_{\varepsilon,L}$ (note that this property depends upon both ε and L) to be the property of real numbers y,

$$D_{\varepsilon,L} : \textit{'the distance from } y \textit{ to } L \textit{ is less than } \varepsilon\textit{'}$$

Definition 2.1. Let (a_n) be a sequence.

We say that (a_n) is a **convergent sequence with limit** L if, for any $\epsilon > 0$, the property $P_{\epsilon,L}$ holds for (a_n) as $n \to \infty$.

Equivalently,

We say that (a_n) is a **convergent sequence with limit** L if, for any $\epsilon > 0$, there is some natural number N such that

$$n \geq N \implies |a_n - L| < \varepsilon.$$

If (a_n) is *not convergent* then we say that (a_n) is **divergent**.

Remark 2.2. 1. Recall that, for real numbers x, y, the non-negative real number $|x-y|$ is the (unsigned) distance between x and y. Thus, the mathematical definition given above is to be read as '*(a_n) is convergent with limit L if, for any $\varepsilon > 0$, the property $D_{\varepsilon,L}$ holds for (a_n) as $n \to \infty$.*'

2. If (a_n) is convergent with limit L then we write

$$\lim_{n\to\infty} a_n = L, \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty.$$

3. Suppose that $f(x)$ is a function defined for all $1 \leq x \leq \infty$. If (a_n) is a sequence so that $a_n = f(n)$, for $n = 1, 2, 3, \ldots$, then $\lim_{n\to\infty} a_n = L$ precisely whenever $\lim_{x\to\infty} f(x) = L$.

4. In this class, the adjective *divergent* is synonymous with *not convergent.*

CHECK YOUR UNDERSTANDING

1. Which of the sequences (a_n), (b_n), (c_n) above are convergent? For those that are convergent, what do you think the limits are?

   (a) (a_n):

   convergent [✓ divergent]

   (if applicable) $L =$ NA

   (b) (b_n):

   [✓ convergent] divergent

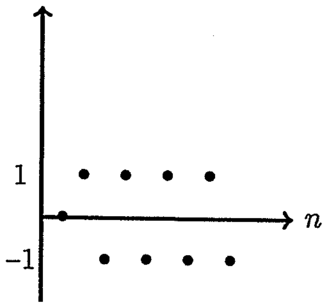
   (if applicable) $L =$ 1

   (c) (c_n):

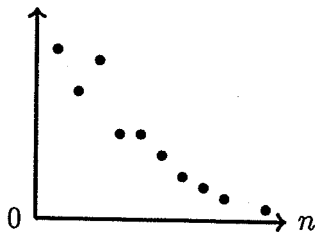
   [✓ convergent] divergent

   (if applicable) $L =$ 0

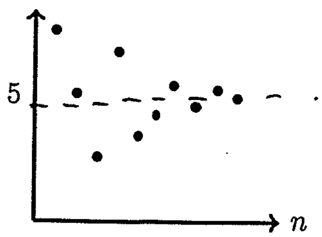
2. Consider the graphs of the following sequences. Identify those sequences that are convergent and, if possible, their limits.



~~Convergent~~
Divergent



Convergent
L=0



Convergent
L=0

Example 2.3. Consider the sequence (a_n), where $a_n = \frac{1}{n}$. We will show directly that (a_n) is convergent with limit $L = 0$.

Suppose we are given some fixed $\varepsilon > 0$. To verify that (a_n) satisfies the statement of Definition 2.1 we have to find an N such that, for each $n \geq$N, we necessarily have

$$|a_n - 0| = \left|\frac{1}{n}\right| < \varepsilon.$$

Observe that, since $a_n > 0$ for all $n = 1, 2, 3, \ldots$, we have $|a_n| = a_n$. Hence, we need N so that, for each $n \geq N$, we necessarily have

$$\frac{1}{n} = a_n = |a_n| < \varepsilon.$$

Rearranging the above inequality, if we take a natural number $N > \frac{1}{\varepsilon}$ then

$$n \geq N \quad \Longrightarrow \quad n \geq N > \frac{1}{\varepsilon} \quad \Longrightarrow \quad |a_n| = \frac{1}{n} < \varepsilon.$$

Thus, we have shown *directly* that $\lim_{n\to\infty} \frac{1}{n} = 0$ (i.e. that (a_n) satisfies the condition required of Definition 2.1).

Corresponding Table:

Limit Laws for Sequences: Let (a_n), (b_n) be convergent sequences, c a constant. Then,

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$,
2. $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$,
3. $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$,
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, whenever $\lim_{n\to\infty} b_n \neq 0$,
5. $\lim_{n\to\infty} a_n^r = (\lim_{n\to\infty} a_n)^r$, if $r > 0$ and $a_n > 0$, for $n = 1, 2, 3, \ldots$.
6. Let $f(x)$ be a continuous function whose domain contains $\{a_n\}_{n\geq 1}$. Then,

$$f(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} f(a_n)$$