Middlebury College

Calculus II: Spring 2018

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February 15 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.1.
- *Calculus*, Spivak, 3rd Ed.: Section 22.
- *AP Calculus BC*, Khan Academy: Infinite sequences.

Sequences: an introduction

1 The phrase 'as n goes to infinity'

Definition 1.1. Let $f(n)$ be a real-valued function, where n is a variable assigned natural numbers only. Let P be a property. We say that **property P holds for $f(n)$ as n goes to infinity** if $f(n)$ satisfies Condition (I) for property P. We will often write **property P holds for $f(n)$ as $n \to \infty$.**

As a rigorous mathematical statement we have the following:

> property P holds for $f(n)$ as $n \to \infty$ if there exists a natural number N such that, for every $n \geq N$, property P holds for $f(n)$.

This means that when we form the table

n	1	2	3	4	5	6	$\cdots$	N	$N+1$	$\cdots$
$f(n)$	$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$\cdots$	$f(N)$	$f(N+1)$	$\cdots$
T/F	*	*	*	*	*	*	$\cdots$	T	T	$\rightarrow$

we can go far enough out to the right (i.e. *there exists a natural number N*) and find a place where there are only Ts in front of us (*for every $n \geq N$, property P holds for $f(n)$*).

2 Sequences The visual representation we have seen for real-valued functions $f(n)$ leads us to the following Definition.

Definition 2.1. Let $f(n)$ be a real-valued function, where n is a variable assigned natural numbers only. The (ordered) collection of all outputs of the function $f(n)$ is called a **sequence.**

A sequence should be considered as an infinitely long list:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & \cdots & n & \cdots \\ f(1) & f(2) & f(3) & f(4) & \cdots & f(n) & \cdots \end{array}$$

We will frequently denote a sequence

$$(a_n)_{n\geq 1} = (a_1, a_2, a_3, a_4, \ldots, a_n, \ldots)$$

where $a_n = f(n)$. In particular, *we care about how we order the outputs of $f(n)$.* We will call a_n the n^{th} **term of the sequence.**

Example 2.2. 1. Let $f(n) = n^2 - 1$. Then, the corresponding sequence is

$$(0, 3, 8, 15, 24, 35, \ldots)$$

2. Let $f(n) = \cos(n)$. Then, the corresponding sequence is

$$(\cos(1), \cos(2), \cos(3), \cos(4), \ldots)$$

> **Remark 2.3.** (a) Sequences will be denoted $(a_n)_{n\geq 1}$, or simply (a_n), where we assume *implicitly* that $a_n = f(n)$ for some real-valued function f whose domain is $\mathbb{N}$.
>
> (b) Given a sequence (a_n) such that $a_n = f(n)$, it is often useful to visualise the graph of $f(n)$ (in a similar manner as the above exercise). We will also call the graph of $f(n)$ **the graph of** (a_n).
>
> (c) Identifying a sequence (a_n) with a real-valued function $f(n)$ allows us to make sense of the following statement, where P is a property of real numbers:
>
> $$\boxed{\text{property } P \text{ holds for } (a_n) \text{ as } n \to \infty.}$$

3. Sequences can be defined **recursively** - this means the n^{th} term is defined as a function of previous terms. For example, the *Fibonacci sequence* (f_n) is defined as follows:

$$f_1 = f_2 = 1, \; f_n = f_{n-1} + f_{n-2}, \quad n \geq 3$$

The first few terms of (f_n) are

$$1, 1, 2, 3, \underline{\;5\;}, \underline{\;8\;}, \underline{\;13\;}, \underline{\;21\;}, \ldots$$

4. Sequences can be defined **without nice formulas**: for example, $a_n = p_n = n^{th}$ prime number[1]. The sequence is

$$(2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots)$$

Recently (January 2018), the largest prime number was discovered: it has 23,249,425 digits and is the natural number

$$467333\ldots - \textit{23,249,413 digits missing} - \ldots 179071$$

Despite knowing that this number is prime, we do not (yet) know where it would appear in the above sequence.

5. Sequences can be defined using **(seemingly) ugly formulas**: for example, a_n = area of $K(n)$, the n^{th} Koch snowflake.

[1]**Note:** there is no known formula for prime numbers. If you can find one, and prove it is correct, then I will give you an A (and you will also be granted a PhD, and a Professorship at Harvard/Princeton/MIT).

Check your understanding

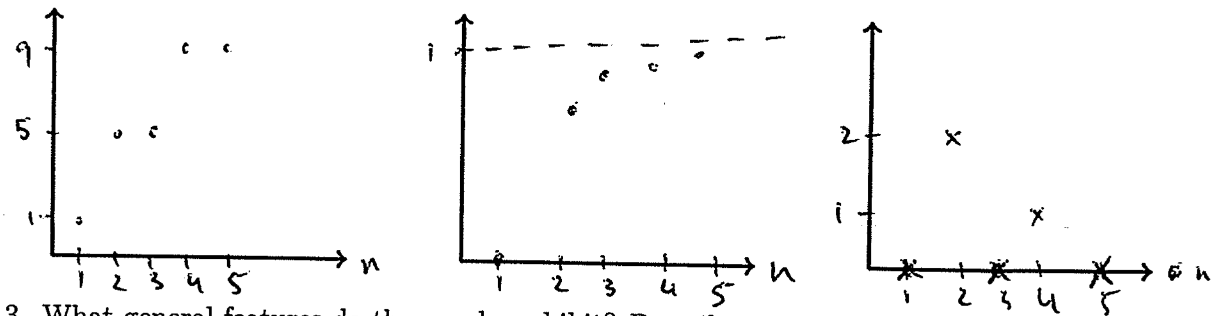
Consider the following real-valued functions

$$g(n) = 2n + (-1)^n, \quad h(n) = 1 - \frac{1}{n^2}, \quad k(n) = 2\left(\frac{1+(-1)^n}{n}\right).$$

1. Write down the first five terms of the corresponding sequence.

	1	2	3	4	5
$g(n)$	1	5	5	9	9
$h(n)$	0	$3/4$	$8/9$	$15/16$	$24/25$
$k(n)$	0	2	0	1	0

2. Plot the graph of the functions above.



3. What general features do the graphs exhibit? Describe as many as you can.

$g(n)$: increasing, bounded below

$h(n)$: ~~de~~increasing, bounded

$k(n)$: 'alternating', bounded.

4. For each function above, give a property P of real numbers so that P holds for that function as $n \to \infty$.

See Feb. 16 Lecture.

We now introduce some terminology for sequences.

Definition 2.4. Let (a_n) be a sequence.

1. The sequence (a_n) is **increasing** if $a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots$. The sequence (a_n) is **strictly increasing** if $a_1 < a_2 < a_3 < \cdots < a_n < \cdots$

2. The sequence (a_n) is **decreasing** if $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$. The sequence (a_n) is **strictly decreasing** if $a_1 > a_2 > a_3 \cdots > a_n > \cdots$.

3. The sequence (a_n) is **bounded below** if there exists a real number m such that $m \leq a_n$, for every n. The sequence (a_n) is **bounded above** if there exists a real number M such that $a_n \leq M$, for every n. A sequence (a_n) is **bounded** if it is bounded above and below. A sequence (a_n) is **unbounded** if it is not bounded.

Example 2.5. The sequence (a_n), where $a_n = \frac{1}{n}$ is strictly decreasing. This can be shown as follows: we must show that $a_n > a_{n+1}$, for every $n = 1, 2, 3, \ldots$ (i.e. $a_1 > a_2$, $a_2 > a_3, \ldots$ etc.). Indeed,

$$a_n - a_{n+1} = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > 0, \quad \text{for } n \geq 1.$$

These choices are non-unique

Hence, $a_n > a_{n+1}$, whenever $n \geq 1$.

This sequence is bounded below by -1 and bounded above by 10. Hence, the sequence is bounded.

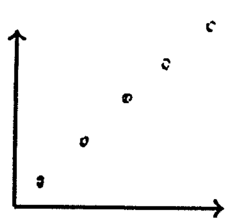
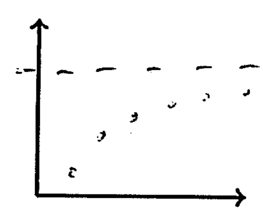
CHECK YOUR UNDERSTANDING

1. For the functions $g(n), k(n)$ given above, state (no need to prove) which of the above attributes - (strictly) increasing/decreasing, (un)bounded - the corresponding sequences possess.

   $g(n)$: increasing, unbounded, bounded below

   $k(n)$: bounded

2. Let (a_n) be an increasing/decreasing sequence. Draw two different possible shapes of the graph of (a_n).

3. Let (a_n) be a bounded sequence. Draw two different possible shapes of the graph of (a_n).