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February 15 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.1.
- *Calculus*, Spivak, 3rd Ed.: Section 22.
- *AP Calculus BC*, Khan Academy: Infinite sequences.

Sequences: an introduction

1 The phrase 'as n goes to infinity'

Definition 1.1. Let $f(n)$ be a real-valued function, where n is a variable assigned natural numbers only. Let P be a property. We say that **property P holds for $f(n)$ as n goes to infinity** if $f(n)$ satisfies Condition (I) for property P. We will often write **property P holds for $f(n)$ as $n \to \infty$**.

As a rigorous mathematical statement we have the following:

$$\boxed{\begin{array}{c}\text{property } P \text{ holds for } f(n) \text{ as } n \to \infty \text{ if there exists a natural} \\ \text{number } N \text{ such that, for every } n \geq N, \text{ property } P \text{ holds for } f(n).\end{array}}$$

This means that when we form the table

n	1	2	3	4	5	6	$\cdots$	N	$N+1$	$\cdots$
$f(n)$	$f(1)$	$f(2)$	$f(3)$	$f(4)$	$f(5)$	$f(6)$	$\cdots$	$f(N)$	$f(N+1)$	$\cdots$
T/F	$*$	$*$	$*$	$*$	$*$	$*$	$\cdots$	T	T	$\rightarrow$

we can go far enough out to the right (i.e. *there exists a natural number N*) and find a place where there are only Ts in front of us (*for every $n \geq N$, property P holds for $f(n)$*).

2 Sequences The visual representation we have seen for real-valued functions $f(n)$ leads us to the following Definition.

Definition 2.1. Let $f(n)$ be a real-valued function, where n is a variable assigned natural numbers only. The (ordered) collection of all outputs of the function $f(n)$ is called a **sequence**.

A sequence should be considered as an infinitely long list:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & \cdots & n & \cdots \\ f(1) & f(2) & f(3) & f(4) & \cdots & f(n) & \cdots \end{array}$$

We will frequently denote a sequence

$$(a_n)_{n \geq 1} = (a_1, a_2, a_3, a_4, \ldots, a_n, \ldots)$$

where $a_n = f(n)$. In particular, we *care about how we <u>order</u> the outputs of $f(n)$*. We will call a_n the n^{th} **term of the sequence**.

Example 2.2. 1. Let $f(n) = n^2 - 1$. Then, the corresponding sequence is

$$(0, 3, 8, 15, 24, 35, \dots)$$

2. Let $f(n) = \cos(n)$. Then, the corresponding sequence is

$$(\cos(1), \cos(2), \cos(3), \cos(4), \dots)$$

Remark 2.3. (a) Sequences will be denoted $(a_n)_{n\geq 1}$, or simply (a_n), where we assume *implicitly* that $a_n = f(n)$ for some real-valued function f whose domain is $\mathbb{N}$.

(b) Given a sequence (a_n) such that $a_n = f(n)$, it is often useful to visualise the graph of $f(n)$ (in a similar manner as the above exercise). We will also call the graph of $f(n)$ **the graph of** (a_n).

(c) Identifying a sequence (a_n) with a real-valued function $f(n)$ allows us to make sense of the following statement, where P is a property of real numbers:

property P holds for (a_n) as $n \to \infty$.

3. Sequences can be defined **recursively** - this means the n^{th} term is defined as a function of previous terms. For example, the *Fibonacci sequence* (f_n) is defined as follows:

$$f_1 = f_2 = 1, \ f_n = f_{n-1} + f_{n-2}, \quad n \geq 3$$

The first few terms of (f_n) are

$$1, 1, 2, 3, \_\_\_\_, \_\_\_\_, \_\_\_\_, \_\_\_\_, \dots$$

4. Sequences can be defined **without nice formulas**: for example, $a_n = p_n = n^{th}$ prime number[1]. The sequence is

$$(2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \dots)$$

Recently (January 2018), the largest prime number was discovered: it has 23,249,425 digits and is the natural number

467333... – *23,249,413 digits missing* – ...179071

Despite knowing that this number is prime, we do not (yet) know where it would appear in the above sequence.

5. Sequences can be defined using **(seemingly) ugly formulas**: for example, a_n = area of $K(n)$, the n^{th} Koch snowflake.

[1]**Note:** there is no known formula for prime numbers. If you can find one, and prove it is correct, then I will give you an A (and you will also be granted a PhD, and a Professorship at Harvard/Princeton/MIT).

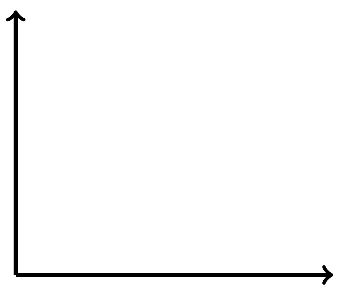
Check your understanding

Consider the following real-valued functions

$$g(n) = 2n + (-1)^n, \quad h(n) = 1 - \frac{1}{n^2}, \quad k(n) = 2\left(\frac{1 + (-1)^n}{n}\right).$$

1. Write down the first five terms of the corresponding sequence.

2. Plot the graph of the functions above.



3. What general features do the graphs exhibit? Describe as many as you can.

4. For each function above, give a property P of real numbers so that P holds for that function as $n \to \infty$.

We now introduce some terminology for sequences.

Definition 2.4. Let (a_n) be a sequence.

1. The sequence (a_n) is **increasing** if $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$. The sequence (a_n) is **strictly increasing** if $a_1 < a_2 < a_3 < \cdots < a_n < \cdots$

2. The sequence (a_n) is **decreasing** if $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$. The sequence (a_n) is **strictly decreasing** if $a_1 > a_2 > a_3 \cdots > a_n > \cdots$.

3. The sequence (a_n) is **bounded below** if there exists a real number m such that $m \leq a_n$, for every n. The sequence (a_n) is **bounded above** if there exists a real number M such that $a_n \leq M$, for every n. A sequence (a_n) is **bounded** if it is bounded above and below. A sequence (a_n) is **unbounded** if it is not bounded.

Example 2.5. The sequence (a_n), where $a_n = \frac{1}{n}$ is strictly decreasing. This can be shown as follows: we must show that $a_n > a_{n+1}$, for every $n = 1, 2, 3, \ldots$ (i.e. $a_1 > a_2$, $a_2 > a_3$,... etc.). Indeed,

$$a_n - a_{n+1} = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} > 0, \quad \text{for } n \geq 1.$$

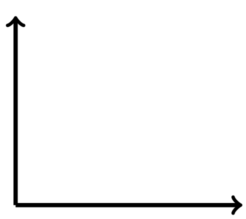
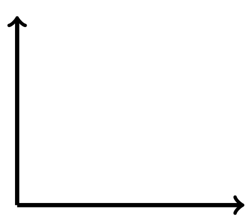
Hence, $a_n > a_{n+1}$, whenever $n \geq 1$.

This sequence is bounded below by _____ and bounded above by _____. Hence, the sequence is ____________________.

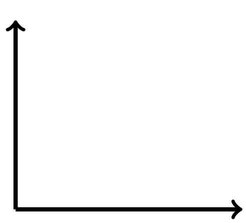
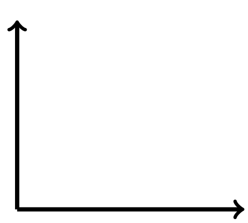
CHECK YOUR UNDERSTANDING

1. For the functions $g(n), k(n)$ given above, state (no need to prove) which of the above attributes - (strictly) increasing/decreasing, (un)bounded - the corresponding sequences possess.

2. Let (a_n) be an increasing/decreasing sequence. Draw two different possible shapes of the graph of (a_n).

3. Let (a_n) be a bounded sequence. Draw two different possible shapes of the graph of (a_n).

3 Limits We introduce the fundamental notion of a **limit** of a sequence. This will build upon our discussion of the phrase 'as $n \to \infty$'.

Consider the increasing and bounded sequence (a_n), where $a_n = 10 - \frac{10}{n}$. Draw the graph of this sequence below and draw the horizontal line $y = 10$.

True/False

1. If P is the property $P : 9 < y < 11$ then P holds for (a_n) as $n \to \infty$.

2. If P is the property P : *'the distance from y to 10 is less than 0.01'* then P holds for (a_n) as $n \to \infty$.

3. If P is the property $P : 9.9999 < y < 10.0001$ then P holds for (a_n) as $n \to \infty$.

4. If P is the property $P : 10 - \frac{1}{2^{100}} < y < 10 + \frac{1}{2^{100}}$ then P holds for (a_n) as $n \to \infty$.

5. Let $\varepsilon > 0$ be any positive real number. If P is the property

$$P : \text{'the distance from } y \text{ to } 10 \text{ is less than } \varepsilon\text{'}$$

then P holds for (a_n) as $n \to \infty$.

We will now formalise the situation observed above.

Let $\varepsilon > 0$ be a positive real number and let L be an arbitrary real number. We define $D_{\varepsilon,L}$ (note that this property depends upon both ε and L) to be the property of real numbers y,

$$D_{\varepsilon,L} : \text{'the distance from } y \text{ to } L \text{ is less than } \varepsilon\text{'}$$

Definition 3.1. Let (a_n) be a sequence. We say that (a_n) is a **convergent sequence with limit** L if, for any $\epsilon > 0$, the property $P_{\epsilon,L}$ holds for (a_n) as $n \to \infty$.

Equivalently, (a_n) is a **convergent sequence with limit** L if, for any $\epsilon > 0$, there is some natural number N such that

$$n \geq N \quad \Longrightarrow \quad |a_n - L| < \varepsilon.$$