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February 14 Lecture

The phrase 'as n tends to infinity'

1 The Natural Numbers. Define the NATURAL NUMBERS to be the collection of all positive integers

$$1, \quad 2, \quad 3, \quad 4, \quad 5, \quad \ldots$$

Denote by $\mathbb{N}$ the collection of natural numbers. The variable n will be used to denote an arbitrary natural number.

2 Functions Recall that, if A and B are sets and $f : A \to B$ is a rule, then we say that f is a **function with domain A and codomain B** if f assigns each element in A to exactly one element in B (i.e. every input has a unique output).

CHECK YOUR UNDERSTANDING

Determine which of the following are functions:

1. A = set of all human mothers in the world, B = set of all humans, f assigns to a mother, a, their children.

   function **not a function**

2. $A = \mathbb{N}$, B = set of real numbers, $f(n) = n^{th}$ decimal digit of π.

   function **not a function**

3. $A = \mathbb{N}$, $B = \{1, -1\}$ the set containing 1 and -1 only, and $f(n) = 1$, if n is even, and $f(n) = -1$ if n is odd.

   function **not a function**

Functions should be familiar objects to you. In this lecture we will begin an investigation in to the behaviour of real-valued functions $f(n)$, having domain $\mathbb{N}$: this means that, to every natural number n we are associating exactly one real number $f(n)$.

3 The phrase "n tends to ∞". Let $f(n)$ be some real-valued function, where n is a natural number. Here are some examples:

Example 3.1. ,

1. $f(n) = p_n$, where p_n is the n^{th} prime number; $f(1) = 2$, $f(2) = 3$, $f(3) = 5$, etc.
2. $f(n) = n^2 + 2$.
3. $f(n) = (-1)^n$; $f(1) = -1$, $f(2) = 1$, $f(3) = -1$, etc.

4. $f(n) = 1 - \frac{1}{n}$; $f(1) = 0$, $f(2) = \frac{1}{2}$, $f(3) = \frac{2}{3}$, $f(4) = \frac{3}{4}$, etc.
5. $f(n) = n^{th}$ decimal digit of π; $f(1) = 1$, $f(2) = 4$, $f(3) = 1$, $f(4) = 5$, etc.
6. $f(n)$ = area of $K(n)$, where $K(n)$ is the n^{th} Koch snowflake (see February 12 Lecture).

Suppose that P is some given property that we wish to check of a real number y: for example, P could be the property

$$\text{'}y > 1\text{'},$$

or the property

'*the difference between* y^2 *and* $-\frac{1}{2}$ *is less than* 0.001'.

We will only consider those properties P that are *well-defined properties*: for a given real number y, either P is true (for y) or P is false (for y).

> **Aim:** given a real-valued function $f(n)$ and a property P, determine those n for which property P is true for $f(n)$.

We could record our results using a table:

n	1	2	3	4	...
$f(n)$	$f(1)$	$f(2)$	$f(3)$	$f(4)$	...
T/F					

For example, let $f(n) = \frac{1}{n}$, P is the property '$y < 0.2$'.

n	1	2	3	4	5	6	7	...
$f(n)$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	...
T/F	F	F	F	F	F	T	T	...

and T hereafter

For any real-valued function $f(n)$ and any property P, exactly one of the following three Conditions must hold:

(I) property P is true for $y = f(n)$, for *all but finitely many n*. — INFINITE Ts, FINITE Fs

(II) property P is false for $y = f(n)$, *except for finitely many n*. — INFINITE Fs, FINITE Ts

(III) neither (I) nor (II). — INFINITE Fs, INFINITE Ts

We will say that $f(n)$ **satisfies Condition (I), (II) or (III) for property** P.

For example, if $f(n) = \frac{1}{n}$ and P : '$y < 0.2$' then $f(n)$ satisfies Condition (I) for property P: property P is true for $y = f(n)$, except when $n = 1, 2, 3, 4, 5$.

Check your understanding

Let P be the property '*y is not an integer*'. Determine the Condition ((I), (II) or (III)) that the following functions satisfy for property P:

1. $f(n) = \frac{p_n}{5}$, where p_n is the n^{th} prime number. Recall that the prime numbers are $2, 3, 5, 7, 11, 13, \ldots$
2. $f(n) = \cos(n\pi)$.
3. $f(n) = 6\left(\frac{1+(-1)^n}{n}\right)$

Hint: it will be useful to write down $f(n)$ for some values of n; try to spot patterns!

Solution:

	n	1	2	3	4	5	6	7	8	9	10	11	12	13
(I)	$P_n/5$	2/5	3/5	1	7/5	11/5	13/5	17/5	19/5	23/5	29/5	31/5	37/5	41/5
	T/F	T	T	F	T	T	T	T	T	T	T	T	T	T
(II)	$\cos(n\pi)$	−1	1	−1	1	−1	1	−1	1	−1	1	−1	1	−1
	T/F	F	F	F	F	F	F	F	F	F	F	F	F	F
(III)	$6\left(\frac{1+(-1)^n}{n}\right)$	0	6	0	3	0	2	0	3/2	0	6/5	0	1	0
	T/F	F	F	F	F	F	F	F	T	F	T	F	F	F

goes FTFT...

We will now consider Condition (I) in more detail. Let P be a property and $f(n)$ a function satisfying Condition (I) for property P. Then, there can exist at most finitely many (possibly zero!) inputs $n_1, \ldots, n_k$ such that property P does not hold for $f(n_1), \ldots, f(n_k)$. *Note:* the inputs $n_1, \ldots, n_k$ are not necessarily the first k natural numbers.

Therefore, if we let N be a natural number that is larger than each of $n_1, \ldots, n_k$, then, for every $n \geq N$, property P holds for $f(n)$.

Example 3.2. Let P be the property '$y > 2$', and let $f(n) = \frac{p_n^2}{49}$, where p_n is the n^{th} prime number. Then, we have

n	1	2	3	4	5	6	7	...
$f(n)$	4/49	9/49	25/49	1	121/49	169/49	289/49	
T/F	F	F	F	F	T	T	T	...

only Ts after here

We see that property P holds for ~~property P~~ f(n) only if $n \geq$ 5. Hence, if we take $N =$ 5 then, for every $n \geq N$, property P holds for $f(n)$. In words, whenever n is greater than or equal to 5, the value $f(n) = \frac{p_n^2}{49} > 2$.

We record our observation below.

To say that a function $f(n)$ satisfies Condition (I) for property P is precisely the same as saying that there exists some large natural number N so that, for every $n \geq N$, property P holds for $f(n)$.

Definition 3.3. Let $f(n)$ be a real-valued function, where n is a variable assigned natural numbers only. Let P be a property. We say that **property P holds for $f(n)$ as n goes to infinity** if $f(n)$ satisfies Condition (I) for property P. We will often write **property P holds for $f(n)$ as $n \to \infty$.**

As a rigorous mathematical statement we have the following:

property P holds for $f(n)$ as $n \to \infty$ if there exists a natural number N such that, for every $n \geq N$, property P holds for $f(n)$.