

February 14 Lecture

The phrase 'as n tends to infinity'

1 The Natural Numbers. Define the NATURAL NUMBERS to be the collection of all positive integers

 $1, 2, 3, 4, 5, \ldots$

Denote by $\mathbb N$ the collection of natural numbers. The variable n will be used to denote an arbitrary natural number.

2 Functions Recall that, if A and B are sets and $f: A \rightarrow B$ is a rule, then we say that f is a function with domain A and codomain B if f assigns each element in A to exactly one element in B (i.e. every input has a unique output).

CHECK YOUR UNDERSTANDING

Determine which of the following are functions:

1. A = set of all human mothers in the world, B = set of all humans, f assigns to a mother, a, their children.

	function	not a function					
2. $A = \mathbb{N}, B = \text{set of real numbers}, f(n) = n^{th}$ decimal digit of π .							
	function	not a function					

3. $A = \mathbb{N}, B = \{1, -1\}$ the set containing 1 and -1 only, and f(n) = 1, if n is even, and f(n) = -1 if n is odd.

function not a function

Functions should be familiar objects to you. In this lecture we will begin an investigation in to the behaviour of real-valued functions f(n), having domain \mathbb{N} : this means that, to every natural number n we are associating exactly one real number f(n).

3 The phrase "*n* tends to ∞ ". Let f(n) be some real-valued function, where *n* is a natural number. Here are some examples:

Example 3.1.,

- 1. $f(n) = p_n$, where p_n is the n^{th} prime number; f(1) = 2, f(2) = 3, f(3) = 5, etc.
- 2. $f(n) = n^2 + 2$.
- 3. $f(n) = (-1)^n$; f(1) = -1, f(2) = 1, f(3) = -1, etc.

- 4. $f(n) = 1 \frac{1}{n}$; f(1) = 0, $f(2) = \frac{1}{2}$, $f(3) = \frac{2}{3}$, $f(4) = \frac{3}{4}$, etc.
- 5. $f(n) = n^{th}$ decimal digit of π ; f(1) = 1, f(2) = 4, f(3) = 1, f(4) = 5, etc.
- 6. f(n) = area of K(n), where K(n) is the n^{th} Koch snowflake (see February 12 Lecture).

Suppose that P is some given property that we wish to check of a real number y: for example, P could be the property

y > 1',

or the property

'the difference between
$$y^2$$
 and $-\frac{1}{2}$ is less than 0.001'

We will only consider those properties P that are well-defined properties: for a given real number y, either P is true (for y) or P is false (for y).

Aim: given a real-valued function f(n) and a property P, determine those n for which property P is true for f(n).

We could record our results using a table:

For example, let $f(n) = \frac{1}{n}$, P is the property 'y < 0.2'.

n	1	2	3	4	5	6	7	•••
f(n)T/F	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	
T/F	F	F	\mathbf{F}	\mathbf{F}	\mathbf{F}	Т	Т	

For any real-valued function f(n) and any property P, exactly one of the following three Conditions must hold:

- (I) property P is true for y = f(n), for all but finitely many n.
- (II) property P is false for y = f(n), except for finitely many n.
- (III) neither (I) nor (II).

We will say that f(n) satisfies Condition (I), (II) or (III) for property P.

CHECK YOUR UNDERSTANDING

Let P be the property 'y is not an integer'. Determine the Condition ((I), (II) or (III)) that the following functions satisfy for property P:

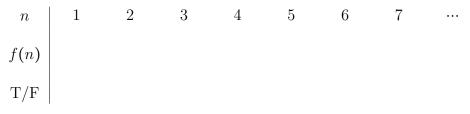
- 1. $f(n) = \frac{p_n}{5}$, where p_n is the n^{th} prime number. Recall that the prime numbers are 2, 3, 5, 7, 11, 13, ...
- 2. $f(n) = \cos(n\pi)$.
- 3. $f(n) = 6\left(\frac{1+(-1)^n}{n}\right)$

Hint: it will be useful to write down f(n) *for some values of* n*; try to spot patterns!* **Solution:**

We will now consider Condition (I) in more detail. Let P be a property and f(n) a function satisfying Condition (I) for property P. Then, there can exist at most finitely many (possibly zero!) inputs n_1, \ldots, n_k such that property P **does not** hold for $f(n_1), \ldots, f(n_k)$. Note: the inputs n_1, \ldots, n_k are not necessarily the first k natural numbers.

Therefore, if we let N be a natural number that is larger than each of n_1, \ldots, n_k , then, for every $n \ge N$, property P holds for f(n).

Example 3.2. Let P be the property 'y > 2', and let $f(n) = \frac{p_n^2}{49}$, where p_n is the n^{th} prime number. Then, we have



We see that property P holds for property P only if n =_____. Hence, if we take N =_____ then, for every $n \ge N$, property P holds for f(n). In words, whenever n is greater than or equal to _____, the value $f(n) = \frac{p_n^2}{49} > 2$.

We record our observation below.

To say that a function f(n) satisfies Condition (I) for property P is precisely the same as saying that there exists some large natural number N so that, for every $n \ge N$, property P holds for f(n).

Definition 3.3. Let f(n) be a real-valued function, where n is a variable assigned natural numbers only. Let P be a property. We say that **property** P holds for f(n) as n goes to infinity if f(n) satisfies Condition (I) for property P. We will often write property P holds for f(n) as $n \to \infty$.

As a rigorous mathematical statement we have the following:

property P holds for f(n) as $n \to \infty$ if there exists a natural number N such that, for every $n \ge N$, property P holds for f(n).