MATH 122B : PRACTICE EXAM III
SOLUTION

1a)

b) F - $\int_1^{\exp(5)} \frac{1}{t}\,dt = \ln(\exp(5)) = 5$

c) F

d) T - DCT with $\sum_{n=1}^{\infty} \frac{2}{n^2+1}$

e) F -

f) F - $y' = \frac{3x}{y} \Rightarrow \int y\,dy = \int 3x\,dx \Rightarrow \frac{y^2}{2} = \frac{3}{2}x^2 + C$

Hyperbola.

g) F - improper integral

$$\int_0^2 \frac{1}{(1-x)^2}\,dx = \underbrace{\int_0^1 \frac{1}{(1-x)^2}}_{\text{DNE}} + \underbrace{\int_1^2 \frac{1}{(1-x)^2}\,dx}_{\text{DNE.}}$$

h) F

(i) F - $a_n = \frac{20}{n}$ $\quad a_1 = 20$ but

$\lim a_n = 0$ $\quad |a_1 - 0| = 20 \not< 10^{-1}$

~~(j)~~

(j) T - integration by parts.

2a) $\int \arcsin(x)\,dx$

$f = \arcsin(x) \qquad g' = 1$

$f' = \dfrac{1}{\sqrt{1-x^2}} \qquad g = x$

$= x\arcsin(x) - \int \dfrac{x}{\sqrt{1-x^2}}\,dx$

$u = 1-x^2$

$du = -2x\,dx$

$= x\arcsin(x) + \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$

$= x\arcsin(x) + \frac{2}{2}\sqrt{1-x^2} + C$

b) ~~Bet~~ Complete square:

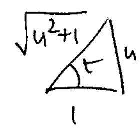
$x^2+4x+5 = (x+2)^2+1$

$\int \sqrt{x^2+4x+5}\,dx = \int \sqrt{(x+2)^2+1}\,dx$

$u = x+2$

$du = dx$

$= \int \sqrt{u^2+1}\,du$



$u = \tan(t)$

$\frac{du}{dt} = \sec^2(t)$

$= \int \sec(t)\cdot\sec^2(t)\,dt$

$= \int \sec^3(t)\,dt$

I will give you formula on exam. Sorry.

$\longrightarrow = \frac{1}{2}\sec(t)\tan(t) + \frac{1}{2}\ln|\sec(t)+\tan(t)| + C$

$= \frac{1}{2}u\sqrt{u^2+1} + \frac{1}{2}\ln\left|\sqrt{u^2+1}+u\right| + C$

$= \frac{1}{2}(x+2)\sqrt{(x+2)^2+1} + \frac{1}{2}\ln\left|\sqrt{(x+2)^2+1}+(x+2)\right| + C.$

3a)
$$\frac{x+2}{25x^2-x^4} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{5-x} + \frac{D}{5+x}$$

$$\Rightarrow \quad x+2 = Ax(25-x^2) + B(25-x^2) + Cx^2(5+x) + Dx^2(5-x)$$

Input	LHS	RHS	
$x=0$	2	$= 25B$	$\Rightarrow B = \frac{25}{2}$
$x=5$	7	$= 250C$	$\Rightarrow C = \frac{7}{250}$
$x=-5$	-3	$= -250D$	$\Rightarrow D = \frac{3}{250}$
$x=1$	3	$= 24A + \frac{24 \cdot 25}{2} + \frac{42}{250} + \frac{12}{250}$	

$$= 24A + 300 + \frac{56}{250}$$

$$\Rightarrow \quad -24A = 297 + \frac{28}{125}$$

$$\Rightarrow \quad A = -\left(\frac{99}{8} + \frac{7}{750}\right)$$

$$\frac{x+2}{25x^2-x^4} = -\left(\frac{99}{8} + \frac{7}{750}\right)\frac{1}{x} + \frac{25}{2x^2} + \frac{7}{250(5-x)} + \frac{3}{250(5+x)}$$

$$\Rightarrow \int \frac{x+2}{25x^2-x^4}\,dx = \int -\left(\frac{99}{8} + \frac{7}{750}\right)\frac{1}{x} + \frac{25}{2}\frac{1}{x^2} + \frac{7}{250(5-x)} + \frac{3}{250(5+x)}\,dx$$

$$= -\left(\frac{99}{8} + \frac{7}{750}\right)\ln|x| - \frac{25}{2}\frac{1}{x} - \frac{7}{250}\ln|5-x| \quad (*)$$

$$+ \frac{3}{250}\ln|5+x| + C.$$

b) Type II integral:

$$\int_1^5 \frac{x+2}{25x^2-x^4}\,dx = \lim_{a\to 5} \int_1^a \frac{x+2}{25x^2-x^4}\,dx$$

$$= \lim_{a\to 5^-} \Big[\ \ast\ \Big]_1^a$$

$$= \lim_{a\to 5} \left[\left(\frac{99}{8} + \frac{7}{750}\right) \ln|a| - \frac{25}{2}\frac{1}{a} - \underbrace{\frac{7}{250}\ln|5-a|}_{\text{DNE}} + \ldots \right]$$

Hence, integral is divergent.

4a) $$y' - \frac{1}{x}y = \frac{x}{\sqrt{x^2+1}}$$

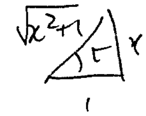
$P(x) = -\frac{1}{x}$, $\int P(x)\,dx = -\ln(x)$

$e^{\int P(x)dx} = I(x) = \frac{1}{x}$

$$\int I(x)\,Q(x)\,dx = \int \frac{1}{\sqrt{x^2+1}}\,dx$$

$x = \tan(t)$

$dx = \sec^2(t)\,dt$



$$= \int \frac{1}{\sec(t)} \cdot \sec^2(t)\,dt$$

$$= \int \sec(t)\,dt$$

$$= \ln|\sec(t) + \tan(t)| + C$$

$$= \ln\left|\sqrt{x^2+1} + x\right| + C$$

$$\Rightarrow \boxed{y = x\ln\left|\sqrt{x^2+1} + x\right| + xC}$$

b) $y' = y^2 \tan(x)$ $\boxed{\text{Separable}}$

$$\int \frac{1}{y^2}\,dy = \int \tan(x)\,dx$$

$$\Rightarrow \quad -\frac{1}{y} = \int \frac{\sin(x)}{\cos(x)}\,dx \qquad u = \cos(x)$$

$$du = -\sin(x)\,dx$$

$$= -\int \frac{1}{u}\,du$$

$$\Rightarrow \quad \frac{1}{y} = \ln|\cos(x)| + C$$

$$\Rightarrow \quad y = \frac{1}{\ln(\cos(x)) + C}$$

$$1 = y(0) = \frac{1}{\ln(1) + C} = \frac{1}{C} \Rightarrow C = 1$$

$$\underline{\text{ie}} \quad y = \frac{1}{\ln|\cos(x)| + 1}$$

5a) Let $a_n = \dfrac{2^n (x+1)^n}{n^{1/3}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{2^{n+1}(x+1)^{n+1}}{(n+1)^{1/3}} \cdot \frac{n^{1/3}}{2^n (x+1)^n} \right|$$

$$= \frac{2|x+1| \cdot n^{1/3}}{(n+1)^{1/3}} = 2|x+1| \cdot \frac{1}{(1+1/n)^{1/3}} \to 2|x+1|$$

Hence converges when $2|x+1| < 1$

$$\Rightarrow \quad |x+1| < \tfrac{1}{2}$$

$$\Rightarrow \quad -\frac{3}{2} < x < -\frac{1}{2}$$

Endpoints:

$x = -\frac{1}{2}$: $\sum \frac{1}{n^{1/3}}$ divergent by p-series

$x = -\frac{3}{2}$ $\sum \frac{(-1)^n}{n^{1/3}}$ convergent by AST

$$\Rightarrow \quad \text{domain is } \boxed{-\frac{3}{2} \le x < -\frac{1}{2}}$$

$$\frac{f^{(7)}(-1)}{7!} = \frac{2^7}{7^{1/3}} \Rightarrow f^{(7)}(-1) = 2^7 \cdot \frac{7!}{7^{1/3}}$$

b) Let $a_n = \frac{(2n)!}{(n!)^2} x^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(2(n+1))! \quad x^{n+1}}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)! \; x^n}\right|$$

$$= \frac{(2(n+1)).(2n+1)}{(n+1)^2} \cdot |X|$$

$$= \frac{2(2n+1)}{n+1} |x|$$

$$= 2 \cdot \frac{(2 + 1/n)}{1 + 1/n} |x| \longrightarrow 4|x|$$

Hence, convergent when $|x| < 1/4$

$$\Rightarrow -\frac{1}{4} < x < 1/4$$

6a) $P(n)$: $n^2 - 3n + 4 = 2s$, for some s

Base case: $P(1)$: $1^2 - 3\cdot 1 + 4 = 0 = 2\cdot 0$,

Inductive step: Assume $P(k)$: $k^2 - 3k + 4 = 2s$, for some k, s

Then $(n+1)^2 - 3(n+1) + 4$

$$= n^2 - 3n + 4 + 2n + 1 - 3$$

$$= 2s + 2(n-1)$$

$$= 2(s+n-1)$$

Hence $P(k+1)$.

Therefore, by math. induction $P(n)$, for all n.

b) Let $P(n)$: $\sum_{j=1}^{n+1} j2^j = n2^{n+2} + 2$

Base: $P(1)$ $\sum_{j=1}^{2} j2^j = 1\cdot 2 + 2\cdot 2^2$

$$= 2 + 8 = 10$$

$$1\cdot 2^3 + 2 = 10$$

Inductive Step: Assume $P(k)$: $\sum_{j=1}^{k+1} j2^j = k2^{k+2} + 2$

Then: $\sum_{j=1}^{k+2} j2^j = \sum_{j=1}^{k+1} j2^j + (k+2)2^{k+2}$

$$= k\cdot 2^{k+2} + 2 + (k+2)2^{k+2}$$

$$= (2k+2)2^{k+2} + 2$$

$$= (k+1)2^{k+3} + 2.$$

Hence $P(k+1)$.

Therefore, by math. induction $P(n)$ for all $n \geq 1$.

7a) Ratio test: $a_n = \dfrac{1.3.5.\dots(2n-1)}{(n!)^2}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\cancel{1}.\cancel{3}.\cancel{5}.\dots(\cancel{2n-1})(2n+1)}{((n+1)!)^2} \cdot \frac{(n!)^2}{\cancel{1}.\cancel{3}.\cancel{5}\dots \cancel{2n-1}}$$

$$= \frac{2n+1}{(n+1)^2} = \frac{2n+1}{n^2+2n+1}$$

$$= \frac{\left(\frac{2}{n}+\frac{1}{n}\right)}{1+\frac{2}{n}+\frac{1}{n^2}} \to \frac{0}{1} = 0$$

Hence, since $\lim \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$ the series is convergent, by Ratio Test.

b) i) Let $P(n)$: $a_n > \frac{1}{3}$

$P(1)$: $a_1 = 1 > \frac{1}{3}$

Assume $P(k)$: $a_k > \frac{1}{3}$, for some k.

Then, $a_{k+1} = \dfrac{a_k+1}{4} > \dfrac{\frac{1}{3}+1}{4} = \dfrac{4}{3}\cdot\dfrac{1}{4} = \frac{1}{3}$

Hence, $P(k+1)$.

$\Rightarrow$ By math. induction $P(n)$, for all n

(ii) Let $P(n)$: $a_{n+1} - a_n < 0$,

$P(1)$: $a_2 = \dfrac{a_1+1}{4} = \dfrac{1+1}{4} = \frac{1}{2} < 1 = a_1$ ✓

Assume $P(k)$, for some k.

Then $a_{k+2} - a_{k+1} = \dfrac{a_{k+1}+1}{4} - \left(\dfrac{a_k+1}{4}\right)$

$= \frac{1}{4}(a_{k+1} - a_k) < 0$, by inductive hypothesis.

Hence, $P(k+1)$

$\Rightarrow$ By math. induction $P(n)$ for all n.

iii) Since (a_n) decreasing, bounded below

$\rightarrow$ (a_n) convergent by MBT

Let $L = \lim a_n$

Then, $L = \lim a_{n+1} = \lim \left(\frac{a_n + 1}{4}\right)$

$= \frac{1}{4} \lim a_n + \frac{1}{4}$

$\Rightarrow L = \frac{1}{4}L + 1/4$

$\Rightarrow \underline{L = 1/3.}$

8a) $f'(x) = 2\cos(x+1)$

$f''(x) = -2\sin(x+1)$

$f'''(x) = -2\cos(x+1)$

$f''''(x) = 2\sin(x+1)$

$\Rightarrow f^{(n)}(-1) = \begin{cases} (-1)^{\frac{n+1}{2}} 2 & n+1 \text{ even} \\ 0 & n \text{ even} \end{cases}$

$\vdots$

$\Rightarrow$ T.S. is

$$f(-1) + \sum_{n=1}^{\infty} \frac{f^{(n)}(-1)}{n!}(x+1)^n$$

$$= 0 + \frac{2}{1!}(x+1) - \frac{2}{3!}(x+1)^3 + \frac{2}{5!}(x+1)^5 + \cdots$$

(b) <u>Note:</u>

$$f(x-1) = 2\sin(x-1+1)$$
$$= 2\sin(x)$$

for all x

$$= 2\cdot\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$\Rightarrow \quad f(x) = 2\underbrace{\left((x+1) - \frac{(x+1)^3}{3!} + \frac{(x+1)^5}{5!} - \cdots\right)}_{\text{T.S. of } f \text{ at } c=-1}$$

Hence, $f(x)$ equals T.S. for all x.