Middlebury College

Calculus II: Spring 2018

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Practice Examination III

You may use the following identities and formulae without proof.

a.

$$\cos^2(x) + \sin^2(x) = 1$$

$$1 + \tan^2(x) = \sec^2(x)$$

b.

$$\begin{aligned} \sin(2x) &= 2\sin(x)\cos(x) \\ \cos(2x) &= \cos^2(x) - \sin^2(x) \\ &= 1 - 2\sin^2(x) \\ &= 2\cos^2(x) - 1 \end{aligned}$$

c.

$$\int \sec(x)dx = \ln(\sec(x) + \tan(x)) + C$$

$$\int \tan(x)dx = \ln|\sec(x)| + C$$

d.

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

e.

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

f.

$$\begin{aligned} \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1} \end{aligned}$$

g.

$$\begin{aligned} \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k} \end{aligned}$$

h.

$$\begin{aligned} \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k}x^k \end{aligned}$$

Instructions:

- You *must* attempt Problem 1.
- Please attempt at least five of Problems 2, 3, 4, 5, 6, 7, 8.
- If you attempt all eight problems then your final score will be the sum of your score for Problem 1 and the scores for the five remaining problems receiving the highest number points.
- Calculators are not permitted.

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1. (10 points) True/False:

   (a) The function $f(x) = 2 - 4x - x^2$, $-1 \leq x \leq 1$ is one-to-one.

(b) Let $e = \exp(1)$ be Euler's number. Then,

$$\int_1^{\exp(5)} \frac{1}{t} dt = e^5$$

(c) The partial fraction decomposition of $\frac{2x+3}{x^2(x+1)}$ is

$$\frac{2}{x} - \frac{3}{x^2} + \frac{1}{2(x+1)}$$

(d) The series $\sum_{n=1}^{\infty} \frac{\cos(n)+1}{n^2+1}$ is convergent.

(e) The function $y = \frac{1}{1+x}$ is a solution to the differential equation $y' - y = xy$.

(f) Let $f(x)$ be a solution to the differential equation $f'(x) = 3x/f(x)$. Then, the set of points $(x, f(x))$ describes part of an ellipse.

(g)

$$\int_0^2 \frac{1}{(1-x)^2} dx = -2$$

(h) The series $\sum_{n=3}^{\infty} \frac{4^{n+1}}{7^{n-1}}$ is divergent.

(i) Let (a_n) be a convergent sequence with limit L. Then, for every n, $|a_n - L| < 10^{-n}$.

(j)

$$\int e^{-x^2} dx = xe^{-x^2} + \int 2x^2 e^{-x^2} dx$$

2. Determine the following antiderivative problems.

(a)

$$\int \arcsin(x) dx$$

(b)

$$\int \sqrt{x^2 + 4x + 5} dx$$

3. (a) Using the method of partial fractions, determine the antiderivative problem

$$\int \frac{x+2}{25x^2 - x^4} dx$$

(b) Determine whether the following improper integral is convergent or divergent

$$\int_1^5 \frac{x+2}{25x^2 - x^4} dx$$

4. (a) Find the general solution to the differential equation.

$$xy' - y - \frac{x^2}{\sqrt{x^2+1}} = 0$$

(b) Determine the unique solution to the following differential equation satisfying $y(0) = 1$

$$\cos(x) y' = y^2 \sin(x)$$

5. (a) Determine the domain of the function $f(x)$ given by the power series

$$f(x) = \sum_{n=1}^{\infty} \frac{2^n (x+1)^n}{n^{1/3}}$$

What is $f^{(7)}(-1)$?

(b) Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$$

6. (a) Using induction show that $n^2 - 3n + 4$ is even, for every natural number n. Recall that an integer r is even if $r = 2s$, for some integer s.

(b) Using induction show that

$$\sum_{j=1}^{n+1} j 2^j = n 2^{n+2} + 2, \qquad n \geq 1$$

7. (a) Determine whether the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(n!)^2}$$

(b) Let (a_n) be the sequence defined recursively by the relation

$$a_n = \frac{a_{n-1} + 1}{4}, \quad a_1 = 1$$

i. Using induction show that $a_n > \frac{1}{3}$, for all $n \geq 1$.
ii. Using induction show that $a_{n+1} - a_n < 0$, for all $n \geq 1$.
iii. Show that (a_n) is convergent and determine $L = \lim_{n \to \infty} a_n$.

8. (a) (10 points) Determine the Taylor series of $f(x) = 2\sin(x+1)$ centred at $c = -1$.

(b) (10 points) Determine those x for which the Taylor series you obtained in part (a) equals $f(x)$.