Middlebury College

Math 122B: Spring 2018
EXAMINATION II

READ THE FOLLOWING INSTRUCTIONS CAREFULLY

DO NOT OPEN THIS PACKET UNTIL INSTRUCTED

Instructions:

- Sign the Honor Code Pledge below.
- Write your name on this exam and any extra sheets you hand in.
- You will have 120 minutes to complete this Examination.
- You **must** attempt Problem 1.
- You **must** attempt **at least three** of Problems 2, 3, 4, 5.
- Your final score will be the sum of your score for Problem 1 and the highest possible score obtained from three of the four remaining problems.
- There are **4 blank pages** attached for scratchwork and/or extra space for solutions.
- **Calculators are not permitted.**
- **Explain your answers *clearly* and *neatly* and in *complete English sentences*.**
- State all Theorems you have used from class. To receive full credit you will need to justify each of your calculations and deductions coherently and fully.
- Correct answers without appropriate justification will be treated with great skepticism.

QUESTION 1:	10 /10
QUESTION 2:	20 /20
QUESTION 3:	20 /20
QUESTION 4:	20 /20
QUESTION 5:	20 /20
TOTAL:	70 /70

NAME: A. L. CAUCHY

"I have neither given nor received unauthorized aid on this assignment"

1. (10 points) True/False (no justification required)

(a) If $f(x)$, $1 \le x \le 2$, admits an inverse function then $g(x) = f(x) + c$, $1 \le x \le 2$, admits an inverse function, for any constant c.

(b)

$$\int_1^{32} \frac{1}{t} dt = \int_1^{16} \frac{1}{t} dt + \int_1^{2} \frac{1}{t} dt$$

(c) If a power series is infinitely differentiable then its radius of convergence is infinite.

(d) If $\sum_{n=0}^{\infty} \frac{x^n}{2^n(n+1)}$ is the Taylor series (centred at $c = 0$) associated to an infinitely differentiable function $f(x)$ then $f'(0) = \frac{1}{4}$.

(e) If the power series $\sum_{n=1}^{\infty} c_n(x+5)^n$ converges at $x = -5$ then it converges at $x = 5$.

Solution: Write T(rue) or F(alse) in the corresponding box below

a) T	b) T	c) F	d) T	e) F

2. Determine the interval of convergence of the following series.

(a) (10 points)

$$\sum_{n=1}^{\infty} \frac{(-3)^n(x+1)^n}{n^2+1}$$

Let $a_n = \dfrac{(-3)^n (x+1)^n}{n^2+1}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{n+1}\cdot(x+1)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{3^n(x+1)^n}\right| = \frac{3|x+1|\,n^2+1}{(n+1)^2+1} \to 3|x+1|$$

Convergent when $|x+1| < 1/3 \Rightarrow -\frac{4}{3} < x < -2/3$

Endpoints:

$\underline{x = -4/3}$: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ convergent, DCT with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\underline{x = -\frac{2}{3}}$: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ convergent, by AST

$\Rightarrow$ $\boxed{-\frac{4}{3} \le x \le -\frac{1}{3}}$

(b) (10 points)

$$\sum_{n=1}^{\infty} \frac{n!}{2^n}(x-8)^n$$

Let $a_n = \dfrac{n!}{2^n}(x-8)^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!\,(x-8)^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n!\,(x-8)^n}\right| = \frac{n+1}{2}|x-8| \longrightarrow +\infty$$

Hence, power series converges at $x=8$ only.

3. (a) (10 points) Using induction show that

$$\sum_{i=1}^{n} i \times (i!) = 1 \times (1!) + 2 \times (2!) + 3 \times (3!) + \ldots + n \times (n!) = (n+1)! - 1$$

for every natural number n.

$P(n): \sum_{i=1}^{n} i \times (i!) = (n+1)! - 1$

$P(1): 1 \times 1! = 1 = (1+1)! - 1$ ✓

Assume $P(k): \sum_{i=1}^{k} i \times (i!) = (k+1)! - 1$

Then,

$$\sum_{i=1}^{k+1} i \times (i!) = \sum_{i=1}^{k} i \times (i!) + (k+1) \times (k+1)!$$
$$= (k+1)! - 1 + (k+1) \cdot (k+1)!$$
$$= (k+1)!(1 + k + 1) - 1 = (k+2)! - 1$$

Hence, $P(k+1) \Rightarrow P(n)$, for all n, by induction.

(b) (10 points) Using induction show that

$$2n + 1 < n^2$$

for $n \geq 3$.

Let $P(n): 2n+1 < n^2$

$P(3): 2.3+1 = 7 < 3^2$ ✓

Assume $P(k): 2k+1 < k^2$
for some k

Then, $(k+1)^2 = k^2 + 2k + 1$
$> 2k+1 + 2k+1$
$> 2k+1+1+1$, since $k \geq 3$
$= 2(k+1) + 1$

Hence, $P(k+1)$.

Therefore, $P(n)$ for all n, by math. induction.

4. Let $f(x)$ be the function defined by the power series

$$\sum_{n=0}^{\infty} \frac{(x-6)^n}{6^n(n+1)^6}$$

(a) (10 points) Determine the domain of $f(x)$.

Let $a_n = \dfrac{(x-6)^n}{6^n(n+1)^6}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-6)^{n+1}}{6^{n+1}(n+2)^6} \cdot \frac{6^n(n+1)^6}{(x-6)^n}\right|$$

$$= \frac{|x-6|}{6} \cdot \frac{(n+1)^6}{(n+2)^6} \longrightarrow \frac{|x-6|}{6}$$

Convergent when $|x-6| < 6 \Rightarrow 0 < x < 12$

Endpoints:

$x=0$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^6}$ convergent by AST

$x=12$: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)^6}$, convergent by DCT with $\sum \frac{1}{n^6}$

$\Rightarrow 0 \leq x \leq 12$

(b) (5 points) What is $f^{(6)}(6)$?

$$\frac{f^{(6)}(6)}{6!} = \frac{1}{6^6(6+1)^6} \Rightarrow f^{(6)}(6) = \frac{6!}{42^6}$$

(c) (5 points) Consider the power series

$$\sum_{n=0}^{\infty} \frac{(x-5)^n}{6^n(n+1)^6}$$

Is this a power series representation for $f(x)$ centred at $c = 5$? Justify your answer.

No! If it were then would require two power series representation to be equal at $x=6$:

$f(6) = 1$, by given power series rep^n above

BUT $\displaystyle\sum_{n=0}^{\infty} \frac{(6-5)^n}{6^n(n+1)^6} = \sum_{n=0}^{\infty} \frac{1}{6^n(n+1)^6} = 1 + \frac{1}{6 \cdot 2^6} + \cdots$

> 1.

5. Let $f(x) = \frac{1}{\sqrt{2x-1}}$.

(a) (10 points) Determine the 4^{th}-degree Taylor polynomial $T_4(x)$ (centred at $c = 1$) associated to $f(x)$.

$$T_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f''''(1)}{4!}(x-1)^4$$

$$f(1) = 1$$

$$f'(x) = -\frac{1}{\cancel{2}}(2x-1)^{-3/2} \cdot \cancel{x} = -\frac{1}{(2x-1)^{3/2}}$$

$$f''(x) = \frac{3}{\cancel{2}} \cdot (2x-1)^{-5/2} \cdot \cancel{x} = \frac{3}{(2x-1)^{5/2}}$$

$$f'''(x) = -\frac{5 \cdot 3}{\cancel{2}} \cdot (2x-1)^{-7/2} \cdot \cancel{x} = \frac{-3 \cdot 5}{(2x-1)^{7/2}}$$

$$f''''(x) = \frac{7 \cdot 5 \cdot 3}{2}(2x-1)^{-9/2} \cdot 2 = \frac{3 \cdot 5 \cdot 7}{(2x-1)^{9/2}}$$

$$\Rightarrow \quad T_4(x) = 1 - (x-1) + \frac{3}{2}(x-1)^2 - \frac{5}{2}(x-1)^3 + \frac{35}{8}(x-1)^4$$

(b) (10 points) Let $F(x)$ be the antiderivative of $f(x)$ satisfying $F(1) = 1$. Write down $T_2(x)$, the 2^{nd}-degree Taylor polynomial (centred at $c = 1$) associated to $F(x)$.

$$T_2(x) = \int 1 - (x-1) + \quad dx$$

$$= (x-1) - \frac{(x-1)^2}{2} + \quad + C$$

Since $F(1) = 1$,

$$1 = T_2(1) = 0 - \frac{0^2}{2} + C$$

$$\Rightarrow \quad T_2(x) = 1 + (x-1) - \frac{(x-1)^2}{2}.$$