Middlebury College

Calculus II: Spring 2018

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Practice Examination II Solution

Instructions:

- You *must* attempt Problem 1.
- Please attempt at least three of Problems 2,3,4,5.
- If you attempt all five problems then your final score will be the sum of your score for Problem 1 and the scores for the three remaining problems receiving the highest number points.
- Calculators are not permitted.

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1. (10 points) True/False:

   (a) $\int_1^{\exp(2x)} \frac{1}{t} dt = 2\int_1^x \frac{1}{t} dt$

   (b) The function $f(x) = 3 - \sin(x+1)$, $-1 \leq x \leq 2$, has an inverse function.

   (c) There exists a power series $\sum_{n=0}^{\infty} c_n (x-3)^n$ that converges at $x = -4$ and diverges at $x = 5$.

   (d) Let $f(x)$ be an infinitely differentiable function with associated Tayloer series (centred at $c = 2$)

   $$\sum_{n=0}^{\infty} \frac{5^n}{n}(x-2)^n.$$

   Then, $f^{(10)}(2) = \frac{5^{10}}{9!}$.

   (e) Let $\sum_{n=0}^{\infty} c_n x^n$ be a power series with interval of convergence $[-2, 2)$. Then, the radius of convergence of $\sum_{n=0}^{\infty} c_n (x-1)^n$ is $R = 1$.

   Solution:

   (a) F

   (b) T

   (c) F

   (d) F

   (e) F

2. Determine the interval of convergence of the following power series.

   (a)

   $$\sum_{n=0}^{\infty} \frac{2^n}{2n+1}(x-3)^n$$

   (b)

   $$\sum_{n=1}^{\infty} \frac{n!}{(2n)!}(x+1)^n$$

Solution:

(a) Let $a_n = \frac{2^n}{2n+1}(x-3)^n$. Then,

$$\left|\frac{a_{n+1}}{a_n}\right| = 2|x-3|\frac{2n+1}{2n+3} \to 2|x-3|$$

Hence, converges when $|x-3| < \frac{1}{2}$ i.e. $\frac{5}{2} < x < \frac{7}{2}$.
Check endpoints:
- $x = \frac{5}{2}$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ convergent by AST.
- $x = \frac{7}{2}$: $\sum_{n=0}^{\infty} \frac{1}{2n+1}$ divergence by Limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$.

Interval of convergence is $\frac{5}{2} \leq x < \frac{7}{2}$.

(b) Let $a_n = \frac{n!}{(2n)!}(x+1)^n$. Then,

$$\left|\frac{a_{n+1}}{a_n}\right| = |x+1|\frac{n+1}{(2n+1)(2n+2)} \to 0$$

Hence, converges for all x.

3. Consider the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(x+2)^n}{2^n(n+2)}$$

(a) Determine the domain A of $f(x)$.

(b) What is $f''(-2)$?

(c) Give a function $g(x)$ with domain A satisfying $\frac{d}{dx}g(x) = f(x)$.

Solution:

(a) Determine interval of convergence: let $a_n = \frac{(x+2)^n}{2^n(n+2)}$. Then,

$$\left|\frac{a_{n+1}}{a_n}\right| = |x+2|\frac{n+2}{2(n+3)} \to \frac{|x+2|}{2}$$

Hence, converges when $|x+2| < 2$ i.e. $-4 < x < 0$, and diverges when $x > 0$ or $x < -4$. Check endpoints:
- $x = -4$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}$ is convergent by AST.
- $x = 0$: $\sum_{n=0}^{\infty} \frac{1}{n+2}$ diverges by LCT with $\sum \frac{1}{n}$.

Hence, domain is $A : -4 \leq x < 2$.

(b) $f(x)$ is function defined by power series so we can differentiate term-by-term: we have

$$f(x) = \frac{1}{2} + \frac{1}{2 \times 3}(x+2) + \frac{1}{4 \times 4}(x+2)^2 + \frac{1}{8 \times 5}(x+2)^3 + \ldots$$

$$\implies f''(x) = \frac{2}{4 \times 4} + \frac{3 \times 2}{8 \times 5}(x+2) + \ldots$$

All higher order terms will vanish when $x = -2$. Hence, $f''(-2) = \frac{2}{4^2} = \frac{1}{8}$.

(c) We integrate $f(x)$ term-by-term:

$$\int f(x)dx = C + \frac{1}{2}(x+2) + \frac{1}{2\times 3}\frac{(x+2)^2}{2} + \frac{1}{4\times 4}\frac{(x+2)^3}{3} + \dots$$

Choose $C = 0$, for example. Let

$$g(x) = \frac{1}{2}(x+2) + \frac{1}{2\times 3}\frac{(x+2)^2}{2} + \frac{1}{4\times 4}\frac{(x+2)^3}{3} + \dots$$

then

$$\frac{d}{dx}(g(x)) = f(x)$$

4. Using induction show that

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \dots + \frac{1}{(2n-1)\times(2n+1)} = \frac{n}{2n+1},$$

for any natural number n.

Solution: Let

$$P(n): \quad \frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \dots + \frac{1}{(2n-1)\times(2n+1)} = \frac{n}{2n+1}$$

Base Case: $n = 1$

$$\frac{1}{1\times 3} = \frac{1}{3} = \frac{1}{2+1}$$

Hence, $P(1)$ is true.

Inductive Step: Assume $P(k)$, for some k,

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \dots + \frac{1}{(2k-1)\times(2k+1)} = \frac{k}{2k+1}$$

Then,

$$\begin{aligned}
&\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \dots + \frac{1}{(2k-1)\times(2k+1)} + \frac{1}{(2(k+1)-1)\times(2(k+1)+1)} \\
=&\frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}, \quad \text{by inductive hyp.} \\
=&\frac{1}{(2k+1)(2k+3)}\left(k(2k+3)+1\right) \\
=&\frac{1}{(2k+1)(2k+3)}(2k^2+3k+1) \\
=&\frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2k+3} = \frac{k+1}{2(k+1)+1}
\end{aligned}$$

Hence, $P(k+1)$. By mathemtical induction, $P(n)$, for every $n \geq 1$.

5. (a) Determine the associated Taylor series (centred at $c = 5$) of $f(x) = \frac{1}{x}$.

(b) Suppose $f(x)$ is a function with associated Taylor series (centred at $c = 0$)

$$\sum_{n=0}^{\infty} \frac{x^n}{n(2n+1)}$$

Determine the associated Taylor series (centred at $c = 0$) of $f'(x)$.

Solution:

(a) We compute

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}, \quad f'''(x) = -\frac{3.2}{x^4}, \quad f''''(x) = \frac{4.3.2}{x^5}$$

Then, $f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}}$. Hence,

$$f^{(n)}(5) = (-1)^n \frac{n!}{5^{n+1}} \qquad \Longrightarrow \qquad \frac{f^{(n)}(5)}{n!} = \frac{(-1)^n}{5^{n+1}}$$

Thus, the Taylor series associated to $f(x) = \frac{1}{x}$ centred at $c = 5$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(5)}{n!}(x-5)^n = \frac{1}{5} + \sum_{n=1}^{\infty} \frac{(-1)^n}{5^{n+1}}(x-5)^n$$

(b) Let $g(x) = f'(x)$. Then, $g^{(n)}(x) = f^{(n+1)}(x)$. Hence,

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{f^{(n+1)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{2n+3}$$

since

$$\frac{f^{(n+1)}(0)}{(n+1)!} = \frac{1}{(n+1)(2(n+1)+1)} = \frac{1}{(n+1)(2n+3)}$$