PRACTICE EXAM I : MATH 122B

1a) F – eg $a_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$

b) T – (s_m) decreasing, bounded

$\Rightarrow (s_m)$ convergent

$\Rightarrow \sum a_n$ convergent

$\Rightarrow (a_n)$ convergent

c) T – MBT

d) T – (s_m) increasing and bounded

$\Rightarrow (s_m)$ convergent

$\Rightarrow \sum a_n$ convergent

e) N/A.

2a) For all n,

$$-1 \le \sin(1/n) \le 1$$

$$\Rightarrow -\frac{1}{2^n} \le \sin\left(\frac{1}{n}\right) \le \frac{1}{2^n}$$

Since $\left(\frac{1}{2^n}\right)$ and $\left(-\frac{1}{2^n}\right)$ converge, by GPT, and $\lim \frac{1}{2^n} = \lim -\frac{1}{2^n} = 0$, we have

$\left(\frac{\sin(1/n)}{2^n}\right)$ convergent to 0.

b) $1 = 2-1 \le 2+\cos(n) \le 2+1 = 3$, for all n.

$$\Rightarrow \quad \frac{n}{2+\cos(n)} \ge \frac{n}{3}$$

$\Rightarrow \quad \frac{n}{2+\cos(n)}$ unbounded as $n \to \infty$, since $\frac{n}{3}$ unbounded

$\Rightarrow \quad \left(\frac{n}{2+\cos(n)}\right)$ divergent by Test for Divergent Sequences.

3) $a_n = \frac{n}{3^n}$

(a) $$a_n - a_{n+1} = \frac{n}{3^n} - \frac{n+1}{3^{n+1}}$$

$$= \frac{1}{3^{n+1}}\left(3n - (n+1)\right)$$

$$= \frac{1}{3^{n+1}}\left(2n-1\right) > 0 \quad , \text{ for } n=1,2,\ldots$$

Hence, $a_n > a_{n+1}$ and (a_n) strictly decreasing.

b) eg • since $a_n > 0$, for all n, lower bound is 0 (any $c \le 0$ will work)

c) Since (a_n) decreasing and bounded below the sequence convergent by MBT.

d) we have $0 < a_n \le \frac{1}{n}$, $n = 1, 2, \ldots$

Since $\lim \frac{1}{n} = 0$, we have, by Squeeze Theorem,

$$\lim_{n \to \infty} a_n = 0.$$

4) Observe, for $n = 1, 2, 3 \ldots$

$$1 = 2 - 1 \le 2 + \sin(n) \le 2 + 1 = 3$$

$$\Rightarrow \quad \frac{2+\sin(n)}{7^n - 2^n} \le \frac{3}{7^n - 2^n}$$

Claim: $\sum_{n=1}^{\infty} \frac{3}{7^n - 2^n}$ convergent

Let $b_n = \frac{1}{7^n}$

$a_n = \frac{3}{7^n - 2^n}$

Then, $\frac{a_n}{b_n} = \frac{3 \cdot 7^n}{7^n - 2^n}$

$$= \frac{7^n}{7^n} \cdot \frac{3}{1 - \left(\frac{2}{7}\right)^n}$$

$$= \frac{3}{1 - \left(\frac{2}{7}\right)^n}$$

$$\longrightarrow \frac{3}{1-0} = 3 > 0$$

by GPT

Hence, since $\sum \frac{1}{7^n}$ convergent (GST), the same is true of $\sum \frac{3}{7^n - 2^n}$.

Finally, by DCT we have $\sum \frac{2+\sin(n)}{7^n - 2^n}$ convergent.

5) For all n,

$$\frac{1}{2} = 2^{-1} \leq 2^{\sin(2n)} \leq 2^{1} = 2$$

$$\Rightarrow \quad \frac{n\,2^{\sin(n)}}{n^3+3n+3} \leq \frac{2n}{n^3+3n+3}$$

<u>Claim</u>: $\sum \frac{2n}{n^3+3n+3}$ convergent.

Let $a_n = \frac{2n}{n^3+3n+3}$, $b_n = \frac{1}{n^2}$

Then, $\frac{a_n}{b_n} = \frac{2n^3}{n^3+3n+3} = \frac{\cancel{n^3}}{\cancel{n^3}} \cdot \frac{2}{1+3/n^2+3/n^3}$

$$\longrightarrow \frac{2}{1+0+0} = 2 > 0.$$

Hence, since $\sum \frac{1}{n^2}$ convergent the same is true of $\sum_{n=1}^{\infty} \frac{2n}{n^3+3n+3}$.

~~Hence~~ Therefore, By DCT, $\sum \frac{n\,2^{\sin(n)}}{n^3+3n+3}$ convergent