Middlebury College

Calculus II: Spring 2018

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Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.8, 11.9
- *Taylor Series*, Integral Calculus, Khan Academy

Keywords: Taylor series, Taylor polynomials, Taylor's theorem

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Taylor's Theorem

Let $f(x)$ be an infinitely differentiable function (i.e. $f^{(n)}(x)$, the n^{th} derivative of $f(x)$ exists, for all n). The Taylor series associated to $f(x)$ (centred at c) is the series

$$\mathcal{T}_c(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots$$

The n^{th}**-degree Taylor polynomial of f centred at c** is

$$T_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n = \sum_{j=0}^{n} \frac{f^{(j)}(c)}{j!}(x-c)^j$$

$$\boxed{\begin{gathered} T_n(x) \text{ is a partial sum of } \mathcal{T}_c(x) \\ \Longrightarrow \quad \lim_{n\to\infty} T_n(x) = \mathcal{T}_c(x) \end{gathered}}$$

Define

$$R_n(x) = f(x) - T_n(x),$$

the n^{th} **remainder** of the Taylor series.

We are interested in understanding when $f(x) = \mathcal{T}_c(x)$. We see that:

$$f(x) = \mathcal{T}_c(x) \quad \Longleftrightarrow \quad \lim_{n\to\infty} R_n(x) = \underline{\ 0\ }$$

Taylor's Theorem provides us with a tool to understand how large the remainder $R_n(x)$ can be:

> **Taylor's Theorem/Inequality**
>
> If $|f^{(n+1)}(x)| \leq M$ for $|x-c| \leq d$ then
>
> $$|R_n(x)| \leq \frac{M}{(n+1)!}|x-c|^{n+1} \leq \frac{Md^{n+1}}{(n+1)!} \quad \text{for } |x-c| \leq d$$

Let's see how to use Taylor's Theorem in practice.

Example: Let $f(x) = \sin(x)$. We showed that the associated Taylor series (centred at $c = 0$) is

$$\mathcal{T}_0(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Since any derivative of $f(x)$ is either equal to $\pm\sin(x)$ or $\pm\cos(x)$, we have

$$|f^{(n)}(x)| \le \overset{M}{1}, \quad \text{for any } n = 0, 1, 2, 3, \ldots, \text{ and any } x.$$

Take, for example, $d = 10$ (this is an arbitrary choice). Then, for any n, we have

$$|f^{(n+1)}(x)| \le 1 \quad \text{whenever } |x| \le 10.$$

Hence, Taylor's Inequality implies that

$$|R_n(x)| \le \frac{M \cdot |x-0|^{n+1}}{(n+1)!} \le \frac{10^{n+1}}{(n+1)!} \quad \text{for } |x| \le 10 \text{ and any } n.$$

This means that

$$-\frac{10^{n+1}}{(n+1)!} \le R_n(x) \le \frac{10^{n+1}}{(n+1)!} \quad \text{for } |x| \le 10.$$

Reminder: For any real number c, $\lim_{n\to\infty} \frac{c^n}{n!} = 0$.

Hence, by the Squeeze Theorem, we conclude that $\lim_{n\to\infty} R_n(x) = 0$, whenever $|x| \le 10$.

Hence, by Taylor's Theorem, for any x in the interval $-10 \le x \le 10$ we have

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

Since our choice $d = 10$ was arbitrary, we have the following power series representation of $\sin(x)$, valid for all x (recall that the angle x must be measured in radians:

$$\boxed{\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}}$$

In your Homework you will show that

$$\boxed{\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \quad \text{for any } x.}$$

Obtaining a power series representation for $f(x)$ allows us to approximate the value of $f(x)$. For example, using the 7^{th} degree Taylor polynomial $T_7(x)$ of $\sin(x)$ (centred at $c = 0$), we compute

$$T_7(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} = 0.841468\ldots$$

This gives the value of $\sin(1)$ correct to four decimal places.

CHECK YOUR UNDERSTANDING

1. Use the Taylor Series for $\cos(x)$ given above to show that the series

$$1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \ldots$$

is convergent and determine its limit. (*You could show this series is convergent using a (tricky) Alternating Series Test argument but this does not give the limit of the series*)

$$-1 = \cos(\pi) = 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \cdots$$

2. Let $a_n = n\sin(1/n)$. Using the Taylor series for $\sin(x)$, show that

$$\lim_{n\to\infty} a_n = 1$$

(*You may have seen how to compute this limit in a previous calculus course using l'Hopital's Rule*)

$$n \cdot \sin(1/n) = n \cdot \left(\frac{1}{n} - \frac{1}{n^3}\cdot\frac{1}{3!} + \frac{1}{n^5}\cdot\frac{1}{5!} - \cdots \right)$$

$$= 1 - \frac{1}{n^2}\cdot\frac{1}{3!} + \frac{1}{n^4}\cdot\frac{1}{5!} - \cdots$$

$$\longrightarrow 1 \text{ as } n \longrightarrow \infty.$$

Bye bye sequences & series