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April 6 Lecture

Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.8, 11.9
- *Taylor Series*, Integral Calculus, Khan Academy

Keywords: Taylor series, Taylor polynomials, Taylor's theorem

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Taylor series

Let $f(x)$ be an infinitely differentiable function (i.e. $f^{(n)}(x)$, the n^{th} derivative of $f(x)$ exists, for all n). The Taylor series associated to $f(x)$ (centred at c) is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + f''(c)(x-c)^2 + f'''(c)(x-c)^3 + \dots$$

Remember: the Taylor series is only **associated** to $f(x)$ - we're not (yet) claiming that it is a valid power series representation of $f(x)$.

Check your understanding

1. Let
$$f(x) = \exp(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$$

   (a) Determine

   $f(0) =$ _____, $f'(0) =$ _____, $f''(0) =$ _____, $f'''(0) =$ _____, $f''''(0) =$ _____

   (b) Determine the first five terms of Taylor series associated to $f(x) = \exp(x)$ at $c = 0$.

(c) Based on your investigations above, what do you think the Taylor series associated to $\exp(x)$ centred at $c = 0$ is?

2. Consider $f(x) = x^3 - 10x^2 + 3x - 2$.

(a) Determine

$$f(2) = \_\_\_\_\_, \quad f'(2) = \_\_\_\_\_, \quad f''(2) = \_\_\_\_\_, \quad f'''(2) = \_\_\_\_\_, \quad f''''(2) = \_\_\_\_\_$$

(b) What is $f^{(n)}(2)$, for $n \geq 5$?

(c) Write down the Taylor series associated to $f(x)$ centred at $c = 2$.

Relationship between $f(x)$ and its Taylor Series

We will now investigate when $f(x)$ is equal to its associated Taylor series (centred at c). For an infinitely differentiable function $f(x)$, we denote its associated Taylor series (centred at c) by

$$\mathcal{T}_c(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + f''(c)(x-c)^2 + \ldots$$

Question: Under what circumstances is a function equal to its associated Taylor series (centred at c) i.e. when is it true that

$$f(x) = \mathcal{T}_c(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{j!}(x-c)^n \text{ ?}$$

Observation: We always have

$$\mathcal{T}_c(c) = f(c) + 0 + 0 + 0 + \ldots = f(c)$$

In particular, $f(x)$ **equals its associated Taylor series at the centre** c.

To better understand how to approach this problem, we recall what it means for the Taylor series to be convergent. We introduce the following terminology:

Taylor polynomials

Let $f(x)$ be an infinitely differentiable function defined on some interval containing $x = c$. The n^{th}**-degree Taylor polynomial of** f **centred at** c is

$$T_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n = \sum_{j=0}^{n} \frac{f^{(j)}(c)}{j!}(x-c)^j$$

For example:

- The 4^{th} degree Taylor polynomial of $f(x) = \exp(x)$ centred at $c = 0$ is

$$T_4(x) = \underline{\hspace{6cm}}$$

- The 2^{nd} degree Taylor polynomial of $f(x) = x^3 - 10x^2 + 3x - 2$ centred at $c = 2$ is

$$T_2(x) = \underline{\hspace{6cm}}$$

We have the following **important observation:**

> $T_n(x)$ is a partial sum of the Taylor series associated to $f(x)$ (centred at c).

Hence, $f(x) = \mathcal{T}_c(x)$ precisely when

$$f(x) = \lim_{n \to \infty} T_n(x) \qquad (*)$$

Define

$$R_n(x) = f(x) - T_n(x),$$

the n^{th} **remainder** of the Taylor series.

We can reinterpret $(*)$: $f(x)$ equals its Taylor series $\mathcal{T}_c(x)$ precisely when

$$\lim_{n \to \infty} R_n(x) = \underline{\hspace{2cm}}$$

Taylor's Theorem provides us with a tool to understand how large the remainder $R_n(x)$ can be:

Taylor's Theorem/Inequality

If $|f^{(n+1)}(x)| \leq M$ for $|x - c| \leq d$ then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-c|^{n+1} \quad \text{for } |x-c| \leq d$$

In particular, whenever $|x - c| \leq d$ we have

$$|R_n(x)| \leq \frac{Md^{n+1}}{(n+1)!}$$

We will see how to use Taylor's Theorem next lecture.