Middlebury College

Calculus II: Spring 2018

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APRIL 5 LECTURE

SUPPLEMENTARY REFERENCES:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.8, 11.9
- *Taylor Series*, Integral Calculus, Khan Academy

KEYWORDS: centre, coefficients, Taylor series

THE CENTRE & COEFFICIENTS; TAYLOR SERIES

In this lecture we will investigate the role of the centre of a power series. Then, we will begin an investigation into the conditions that a function $f(x)$ must satisfy in order for it to possess a power series representation (centred at c): this will lead us to the **Taylor series of** $f(x)$.

The centre

The function $f(x) = \frac{1}{1-x}$ has power series representation

$$f(x) = 1 + \sum_{n=1}^{\infty} x^n, \quad \text{valid whenever } -1 < x < 1$$

Let's do something wacky: we can write

$$f(x) = \frac{1}{1-x} = \frac{1}{2-(x+1)} = \frac{\frac{1}{2} \cdot \frac{1}{}}{1 - \left(\frac{x+1}{2}\right)}$$

Expanding

$$\frac{1}{1-\left(\frac{x+1}{2}\right)} = 1 + \left(\frac{x+1}{2}\right) + \left(\frac{x+1}{2}\right)^2 + \left(\frac{x+1}{2}\right)^3 + \cdots$$

which is valid whenever

$$\left|\frac{x+1}{2}\right| < 1 \iff -3 < x < 1$$

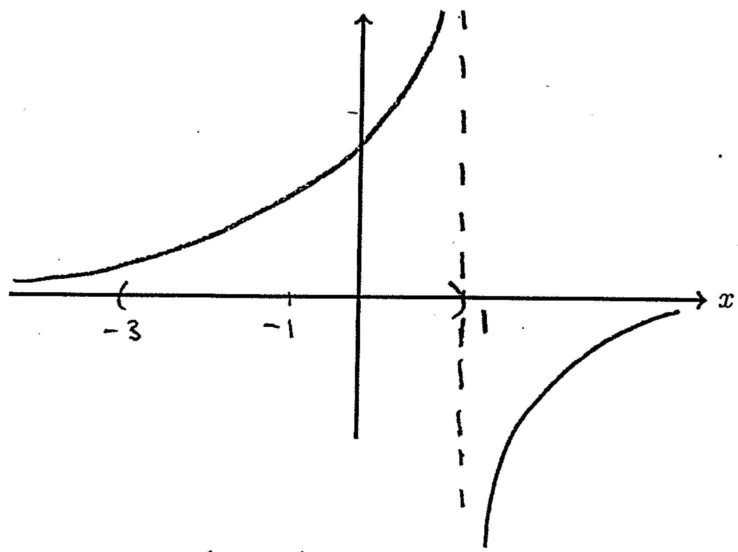
In particular, we've determine a power series represention of $f(x) = \frac{1}{1-x}$ centred at $c = -1$:

$$f(x) = \frac{1}{2} + \frac{1}{4}(x+1) + \frac{1}{8}(x+1)^2 + \cdots$$

valid whenever $-3 < x < 1$

> **Question:**
>
> How do we interpret this power series expansion of $f(x) = \frac{1}{1-x}$ centred at $c = -1$?

Consider the graph of $f(x) = \frac{1}{1-x}$:



Observation:

Shifting centre to left gives power series with larger interval of convergence

CHECK YOUR UNDERSTANDING

1. Based on the above observation, what do you expect to tbe the interval of convergence for a power series representation of $f(x) = \frac{1}{1-x}$ centred at $c = -5$?

$$-11 < x < 1$$

2. Verify your guess by writing down the first four nonzero terms of the power series representation $\sum_{n=0}^{\infty} c_n(x+5)^n$ for $f(x) = \frac{1}{1-x}$ centred at $c = -5$. *Hint: perform a similar computation as above.*

$$\frac{1}{1-x} = \frac{1}{6-(x+5)} = \frac{1}{6} \cdot \frac{1}{1-\left(\frac{x+5}{6}\right)}$$

$$= \frac{1}{6}\left(1 + \left(\frac{x+5}{6}\right) + \left(\frac{x+5}{6}\right)^2 + \left(\frac{x+5}{6}\right)^3 + \cdots\right)$$

$$= \frac{1}{6} + \frac{1}{6^2}(x+5) + \frac{1}{6^3}(x+5)^2 + \cdots$$

Taylor Series

We are interested in the following problem.

> **Problem:** Let $f(x)$ be a given function.
>
> 1. Does $f(x)$ admit a power series representation? If so, how do we determine it?
>
> 2. For which x is the power series representation valid?

Definition: Let $f(x)$ be an infinitely differentiable function. The **Taylor series associated to** $f(x)$ **centred at** c is the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$$

When $c = 0$ the Taylor series associated to $f(x)$ is also called the **Maclaurin series** of $f(x)$ (after the Scottish mathematician Colin Maclaurin (1698-1746)).

> **Remark:** [IMPORTANT!]
>
> At this time, the Taylor series of $f(x)$ centred at c is a series that we are **associating** to $f(x)$; we are **not** saying that the Taylor series is equal to $f(x)$. We will investigate when the Taylor series equals $f(x)$ in the next lecture.

Example:

Let $f(x) = \sin(x)$. Then, $f(x)$ is an infinitely differentiable function and we can determine its associated Taylor series at $c = 0$ (i.e. the Maclaurin series). We compute

$$f(0) = 0$$

$$f'(0) = 1$$

$$f''(0) = 0$$

$$\vdots$$

In general

$$f^{(n)}(0) = \begin{cases} 0, & \text{if } n \text{ even,} \\ (-1)^k, & \text{if } n = 2k+1. \end{cases}$$

Hence, the Taylor series associated to $f(x) = \sin(x)$ at $c = 0$ is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

CHECK YOUR UNDERSTANDING

The coefficients

Recall that a power series $\sum_{n=0}^{\infty} c_n(x-c)^n$ is completely determined by its **centre** and its **coefficients**

> **Question:**
>
> Can we determine the coefficients of the power series representation of $f(x) = \frac{1}{1-x}$ centred at $c = -5$ without playing the calculus/series game?

Thankfully, yes!

MATHEMATICAL WORKOUT

Let $f(x) = \frac{1}{1-x}$.

1. Using induction show that

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}, \quad n = 1, 2, 3, \ldots$$

Here $f^{(n)}(x)$ is the n^{th} derivative of $f(x)$. *Hint:* $f^{(n+1)}(x) = \frac{d}{dx} f^{(n)}(x)$.

BASE CASE:

$n=1$ $\quad f'(x) = \left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$ ✓

INDUCTIVE STEP:

Assume $f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}$, for some k. Then

$$f^{(k+1)}(x) = \left(f^{(k)}(x)\right)' = \left(\frac{k!}{(1-x)^{k+1}}\right)'$$

$$= \frac{(k+1)\,k!}{(1-x)^{k+2}} = \frac{(k+1)!}{(1-x)^{k+2}}$$ ✓

2. Compute the following quantities:

$f(-5) =$ $1/6$, $f'(-5) =$ $1/6^2$, $f''(-5) =$ $2/6^3$, $f'''(-5) =$ $3!/6^4$

3. What is the relationship between the values just computed andthe coefficients c_0, c_1, c_2, c_3 you obtained for the power series representation of $f(x) = \frac{1}{1-x}$ at $c = -5$?

$c_0 = f(-5)$

$c_1 = f'(-5)$

$c_2 = \frac{f''(-5)}{2!}$

$c_3 = \frac{f'''(-5)}{3!}$