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April 5 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.8, 11.9
- *Taylor Series*, Integral Calculus, Khan Academy

Keywords: centre, coefficients, Taylor series

The centre & coefficients; Taylor series

In this lecture we will investigate the role of the centre of a power series. Then, we will begin an investigation into the conditions that a function $f(x)$ must satisfy in order for it to possess a power series representation (centred at c): this will lead us to the **Taylor series of** $f(x)$.

The centre

The function $f(x) = \frac{1}{1-x}$ has power series representation

$$f(x) = 1 + \sum_{n=1}^{\infty} x^n, \quad \text{valid whenever } -1 < x < 1$$

Let's do something wacky: we can write

$$f(x) = \frac{1}{1-x} = \frac{1}{2-(x+1)} = \underline{\hspace{6cm}}$$

Expanding

$$\underline{\hspace{6cm}} = \underline{\hspace{12cm}}$$

which is valid whenever

$$\underline{\hspace{4.5cm}} \iff \underline{\hspace{3cm}} < x < \underline{\hspace{3cm}}$$

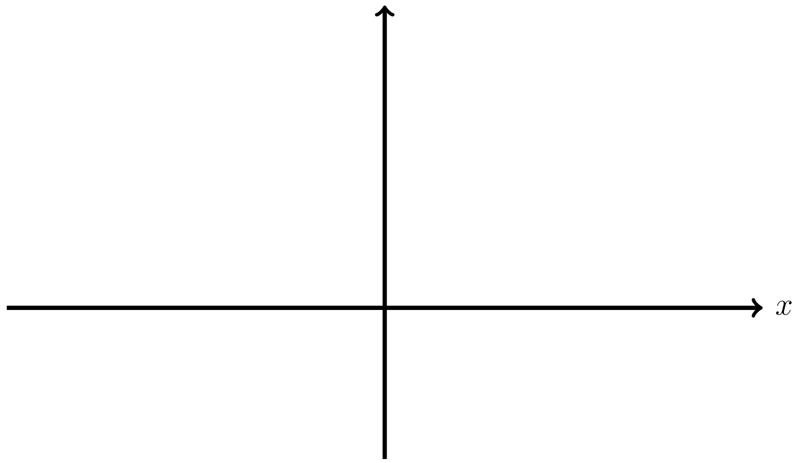
In particular, we've determine a power series represention of $f(x) = \frac{1}{1-x}$ centred at $c = -1$:

$$f(x) = \underline{\hspace{10cm}}$$

valid whenever $\underline{\hspace{3cm}} < x < \underline{\hspace{3cm}}$

> **Question:**
>
> How do we interpret this power series expansion of $f(x) = \frac{1}{1-x}$ centred at $c = -1$?

Consider the graph of $f(x) = \frac{1}{1-x}$:



Observation:

Check your understanding

1. Based on the above observation, what do you expect to tbe the interval of convergence for a power series representation of $f(x) = \frac{1}{1-x}$ centred at $c = -5$?

2. Verify your guess by writing down the first four nonzero terms of the power series representation $\sum_{n=0}^{\infty} c_n (x+5)^n$ for $f(x) = \frac{1}{1-x}$ centred at $c = -5$. *Hint: perform a similar computation as above.*

The coefficients

Recall that a power series $\sum_{n=0}^{\infty} c_n(x-c)^n$ is completely determined by its **centre** and its **coefficients**

> **Question:**
>
> Can we determine the coefficients of the power series representation of $f(x) = \frac{1}{1-x}$ centred at $c = -5$ without playing the calculus/series game?

Thankfully, **yes!**

MATHEMATICAL WORKOUT

Let $f(x) = \frac{1}{1-x}$.

1. Using induction show that

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}, \quad n = 1, 2, 3, \ldots$$

Here $f^{(n)}(x)$ is the n^{th} derivative of $f(x)$. *Hint:* $f^{(n+1)}(x) = \frac{d}{dx} f^{(n)}(x)$.

BASE CASE:

INDUCTIVE STEP:

2. Compute the following quantities:

$f(-5) =$ __________, $f'(-5) =$ __________, $f''(-5) =$ __________, $f'''(-5) =$ __________

3. What is the relationship between the values just computed andthe coefficients c_0, c_1, c_2, c_3 you obtained for the power series representation of $f(x) = \frac{1}{1-x}$ at $c = -5$?

Taylor Series

We are interested in the following problem.

Problem: Let $f(x)$ be a given function.

1. Does $f(x)$ admit a power series representation? If so, how do we determine it?
2. For which x is the power series representation valid?

Definition: Let $f(x)$ be an infinitely differentiable function. The **Taylor series associated to** $f(x)$ **centred at** c is the series

$$\sum_{n=0}^{\infty} \underline{\hspace{3cm}}(x-c)^n = \underline{\hspace{8cm}}$$

When $c = 0$ the Taylor series associated to $f(x)$ is also called the **Maclaurin series of** $f(x)$ (after the Scottish mathematician Colin Maclaurin (1698-1746)).

Remark: [IMPORTANT!]

At this time, the Taylor series of $f(x)$ centred at c is a series that we are **associating** to $f(x)$; we are **not** saying that the Taylor series is equal to $f(x)$. We will investigate when the Taylor series equals $f(x)$ in the next lecture.

Example:
Let $f(x) = \sin(x)$. Then, $f(x)$ is an infinitely differentiable function and we can determine its associated Taylor series at $c = 0$ (i.e. the Maclaurin series). We compute

$$f(0) = \underline{\hspace{1cm}}$$

$$f'(0) = \underline{\hspace{1cm}}$$

$$f''(0) = \underline{\hspace{1cm}}$$

$$\vdots$$

In general

$$f^{(n)}(0) = \begin{cases} \underline{\hspace{1cm}}, & \text{if } n \text{ even,} \\ \\ \underline{\hspace{1cm}}, & \text{if } n = 2k+1. \end{cases}$$

Hence, the Taylor series associated to $f(x) = \sin(x)$ at $c = 0$ is

$$\underline{\hspace{9cm}}$$

Check your understanding

1. Let

$$f(x) = \exp(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$$

(a) Determine

$$f(0) = \_\_\_\_\_, \quad f'(0) = \_\_\_\_\_, \quad f''(0) = \_\_\_\_\_, \quad f'''(0) = \_\_\_\_\_, \quad f''''(0) = \_\_\_\_\_$$

(b) Determine the first five terms of Taylor series associated to $f(x) = \exp(x)$ at $c = 0$.

(c) Based on your investigations above, what do you think the Taylor series associated to $\exp(x)$ centred at $c = 0$ is?

2. Consider $f(x) = x^3 - 10x^2 + 3x - 2$.

(a) Determine

$$f(2) = \_\_\_\_\_, \quad f'(2) = \_\_\_\_\_, \quad f''(2) = \_\_\_\_\_, \quad f'''(2) = \_\_\_\_\_, \quad f''''(2) = \_\_\_\_\_$$

(b) What is $f^{(n)}(2)$, for $n \geq 5$?

(c) Write down the Taylor series associated to $f(x)$ centred at $c = 2$.