Middlebury College

Calculus II: Spring 2018

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# April 4 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.8, 11.9
- *Power Series*, Integral Calculus, Khan Academy

Keywords: power series, term-by-term differentiation, term-by-term integration

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## Power Series II

Recall that a power series is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-c)^n$$

We are interested in determining those $x$ for which the power series is convergent. We have seen that there is a largest interval on which the power series converges. Moreover, there are three possibilities for the **interval of convergence** of a power series:

1. the interval of convergence is a single point $x = c$;
2. the interval of convergence is a finite interval of the form

   $(c-R, c+R)$, or $[c-R, c+R]$, or $(c-R, c+R]$, or $[c-R, c+R)$

   for some $R$ (the **radius of convergence**)
3. the interval of convergence is $(-\infty, \infty)$

On the interval of convergence, the power series gives a well-defined function. Today we will investigate the properties of this function.

### Properties of functions represented by power series

Given a power series $\sum_{n=0}^{\infty} c_n(x-c)^n$ with interval of convergence $I$, we can define a function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-c)^n, \quad \text{for } x \text{ in } I.$$

Any function defined in this way admits the following properties:

**Properties of functions defined by power series:**

- $f(x)$ is differentiable (and therefore continuous) on $I$ and

$$f'(x) = c_1 + 2c_2(x-c) + 3c_3(x-c)^2 + \dots$$

i.e. the power series can be differentiated term-by-term.

- $f(x)$ is integrable and

$$\int f(x)dx = C + c_0(x-c) + \frac{c_1}{2}(x-1)^2 + \frac{c_2}{3}(x-c)^3 + \dots$$

i.e. the power series can be integrated term-by-term.

Moreover, the series obtained by differentiation/integration are centred at $c$ and have the same radius of convergence as $f(x)$. The endpoints of the interval of convergence need to be given further investigation.

These results will be very useful in giving power series representations of well-known functions.

**Example:**

1. Recall that if $|x| < 1$ then

$$1 + \sum_{n=1}^{\infty} x^n = \underline{1 + x + x^2 + \dots} = \underline{\frac{1}{1-x}}$$

Hence, we have

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$= \frac{d}{dx}(\underline{1 + x + x^2 + x^3 + \dots})$$

$$= \underline{1 + 2x + 3x^2 + 4x^3 + \dots}$$

Thus, we have found a power series representation of $f(x) = \frac{1}{(1-x)^2}$ centred at $c = 0$.

2. Consider the function $f(x) = \frac{1}{(1-3x)^2}$. We will determine a power series representation of this function centred at $c = 0$.

First, we record the essential observation:

$$f(x) = \frac{1}{3}\left[\frac{d}{dx}\left(\frac{1}{1-3x}\right)\right]$$

Using geometric series, we find a power series representation

$$\frac{1}{1-3x} = 1 + \sum_{n=1}^{\infty}(3x)^n = \underline{1 + 3x + 9x^2 + 27x^3 + \dots}$$

valid whenever $|x| < \frac{1}{3}$. Hence,

$$\begin{aligned}
f(x) &= \frac{1}{3}\left[\frac{d}{dx}\left(\frac{1}{1-3x}\right)\right] \\
&= \frac{1}{3}\left[\frac{d}{dx}\left(1 + 3x + 9x^2 + 27x^3 + \dots\right)\right] \\
&= \frac{1}{3}\left(3 + 18x + 81x^2 + \dots\right), \quad \text{differentiating term-by-term} \\
&= 1 + 6x + 27x^2 + \dots \\
&= 1 + \sum_{n=1}^{\infty} (n+1) 3^n x^n
\end{aligned}$$

In this case, it can be a bit tricky to write down the coefficients using a formulae. In general, you should try to find a formula for the coefficients (if asked to so) or write down (at least) the first four nonzero terms of the power series

3. Observe that
$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \dots$$
This power series representation is valid whenever $|x| < 1$. Hence,
$$\ln(1+x) = \int \frac{1}{1+x}dx = \int (1 - x + x^2 - x^3 + x^4 - x^5 + \dots)dx$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + C$$
As $0 = \ln(1)$ we find $C = 0$.
The radius of convergence of $\frac{1}{1+x}$ is $R = 1$ (because $\frac{1}{1+x}$ converges when $-1 < x < 1$), and the radius of convergence of the power series expansion of $\ln(1+x)$ is also $R = 1$. However, when $x = 1$ the series is convergent (by Alternating Series Test), and we determine the limit
$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \dots$$

4. The following series expansion of $\arctan(x)$ was first given by James Gregory, a 17th century Scottish mathematician (who lived $< 10$ miles from where I grew up!).
Recall that
$$\arctan(x) = \int \frac{1}{1+x^2}dx$$
Now, since
$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$$
we find
$$\arctan(x) = \int \frac{1}{1+x^2}dx = \int 1 - x^2 + x^4 - x^6 + x^8 - \dots dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Since $\arctan(0) = 0$ we find $C = 0$.

The radius of convergence of $\frac{1}{1+x^2}$ is $R = 1$ so that same is true of the power series representation of $\arctan(x)$. When $x = 1$, the series converges (by Alternating Series Test) so that

$$\frac{\pi}{4} = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

In this way,we obtain a series approximation to $\pi$:

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \ldots$$

However, this series approximation is quite inefficient: we require 100 terms of the above series to obtain a value that's correct to two decimal places; calculating $\pi$ correct to 10 decimal places requires approx. five billion terms!

This approximation was first discovered by the Indian mathematician Madhava of Sangamagrama in the 14th Century, and later rediscovered by Gottfried Leibniz in the 17th Century.

In 1910 the Indian mathematician Srinivisa Ramanujan (at the age of 23) discovered the following series expansion

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}$$

This series can be used to compute approximations to $\pi$ with incredible speed: the first term of this series is

$$\frac{2206\sqrt{2}}{9801} \implies \frac{9801}{2206\sqrt{2}} = 3.14159273001\ldots$$