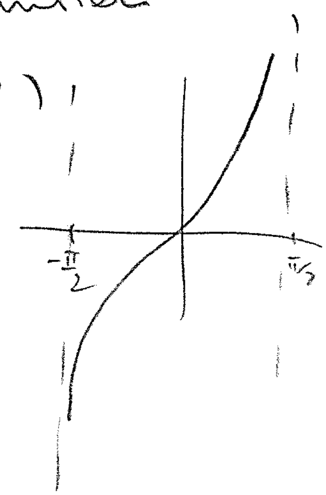
MATH 122: HOMEWORK 4/14

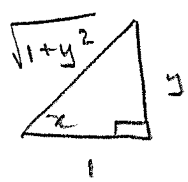
1) $\tan(x)$ is increasing on $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$\Rightarrow$ it admits an inverse function

Since range $\tan(x) = (-\infty, \infty)$, the inverse, $\arctan(y)$, has domain $(-\infty, \infty)$.



$y = \tan(x)$ We have $\arctan(y) = x$.



$$\frac{d}{dy}\arctan(y)$$

$$= \frac{1}{\tan'(\arctan(y))}$$

$$= \frac{1}{\sec^2(\arctan(y))}$$

$$= \cos^2(\arctan(y))$$

$$= \cos^2(x) = \left(\frac{1}{\sqrt{1+y^2}}\right)^2$$

$$= \frac{1}{1+y^2}.$$

2 a) Let $a_n = \frac{(x-3)^n}{n+1}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+2} \cdot \frac{n+1}{(x-3)^n}\right| = |x-3| \frac{n+1}{n+2}$$

$$\longrightarrow |x-3|$$

Require $|x-3| < 1 \Rightarrow 2 < x < 4$

Check endpoints:

$(x=2)$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$ <u>convergent</u> by AST

$(x=4)$ $\sum_{n=1}^{\infty} \frac{1}{n+1}$ divergent

$\Rightarrow$ $\boxed{2 \leq x < 4.}$

b) $\sum_{n=\emptyset}^{\infty} \frac{10^n x^n}{n^3}$ $a_n = \frac{10^n x^n}{n^3}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{10^{n+1} x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{10^n x^n}\right|.$$

$$= 10|x| \cdot \frac{n^3}{(n+1)^3} \longrightarrow 10|x|$$

Require $10|x| < 1$ $\Rightarrow$ $|x| < \frac{1}{10}$

$\Rightarrow$ $-\frac{1}{10} < x < \frac{1}{10}$.

Check endpoints;

<u>$x = -1/10$</u>: $\sum_{n=\emptyset}^{\infty} \frac{(-1)^n}{n^3}$ convergent by AST

$x = 1/10$: $\sum_{n=1}^{\infty} \frac{1}{n^3}$ convergent p-series.

$\Rightarrow$ $-\frac{1}{10} \leq x \leq \frac{1}{10}$.

c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-x)^n}{n^2 2^n}$: Let $a_n = \dfrac{(-x)^n}{n^2 2^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)^2\, 2^{n+1}} \cdot \frac{2^n (n^2)}{x^n}\right|$$

$$= \frac{|x|}{2} \cdot \frac{n^2}{(n+1)^2} \longrightarrow \frac{|x|}{2}$$

Require $\dfrac{|x|}{2} < 1 \Rightarrow |x| < 2$

$\Rightarrow -2 < x < 2$.

<u>Check endpoints:</u>

<u>$x=2$</u>: $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ convergent by AST

<u>$x=-2$</u>: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent by p-series.

$$\boxed{-2 \le x \le 2}$$

d) 4 $\displaystyle\sum_{n=0}^{\infty} n(x+2)^n$ Let $a_n = n(x+2)^n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+2)^{n+1}}{n\,(x+2)^n}\right| = \frac{n+1}{n} \cdot |x+2|$$

$$\longrightarrow |x+2|$$

Require $|x+2| < \Rightarrow -3 < x < -1$

Check endpoints:

$\underline{x = -1}$: $\sum_{n=0}^{\infty} n$ divergent p-series

$\underline{x = -3}$: $\sum_{n=0}^{\infty} (-1)^n n$ divergent by Test for divergence.

e) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$. Let $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x^2)^{n+1}}{4^{n+1}((n+1)!)^2} \cdot \frac{4^n (n!)^2}{(x^2)^n}\right|$$

$$= \frac{|x^2|}{4} \cdot \frac{(n!)^2}{((n+1)!)^2}$$

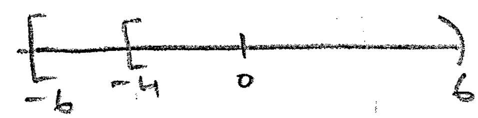
$$= \frac{|x^2|}{4} \cdot \frac{1}{(n+1)^2} \longrightarrow 0$$

$\Rightarrow$ Convergent for all x.

3a) $$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} c_n 1^n$$

ie power series evaluated at $x = 1$

$\Rightarrow$ convergent.



Interval of convergence is at least

$-4 \leq x < 4$

no larger than

$-6 \leq x < 6$.

b) $\sum c_n 8^n$ is power series evaluated at $x = 8$.

Since 8 outside of largest possible I.o.C.

$\Rightarrow$ divergent

c) $$\sum c_n (-3)^{-n} = \sum c_n \left(\frac{-1}{3}\right)^n$$

$\Rightarrow$ convergent

d) $$\sum c_n (-1)^n 7^n = \sum c_n (-7)^n$$

$\Rightarrow$ divergent.

4a) $f(x) = \dfrac{2}{1-3x}$

$= 2\left(1 + 3x + (3x)^2 + (3x)^3 + \cdots\right)$

$= 2 + 3x + 18x^2 + 54x^3 + \cdots$

Valid when $|3x| < 1$

$\Rightarrow |x| < \frac{1}{3}$

$\Rightarrow -\frac{1}{3} < x < \frac{1}{3}$.

b) $f(x) = \dfrac{1}{2+3x}$

$= \dfrac{1}{2} \cdot \dfrac{1}{1 + \frac{3x}{2}}$

$= \dfrac{1}{2}\left(1 - \left(\frac{3x}{2}\right) + \left(\frac{3x}{2}\right)^2 - \left(\frac{3x}{2}\right)^3 + \cdots\right)$

$= \dfrac{1}{2} - \dfrac{3x}{4} + \dfrac{9x^2}{8} - \dfrac{27x^3}{16} + \cdots$

Valid when $\left|\frac{3x}{2}\right| < 1 \Rightarrow |x| < \frac{2}{3}$

$\Rightarrow -\frac{2}{3} < x < \frac{2}{3}$.

c) $f(x) = \dfrac{1}{1+4x^2}$

$= 1 - 4x^2 + (4x^2)^2 - (4x^2)^3 + \cdots$

$= 1 - 4x^2 + 16x^4 - 64x^6 + \cdots$

valid when $|4x^2| < 1$

$\Rightarrow |x^2| < \frac{1}{4} \Rightarrow \boxed{-\frac{1}{2} < x < \frac{1}{2}}$

5a) T

b) T eg $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-1)^n$

c) F , centre must be in I.o.C.

d) T