Middlebury College

Calculus II: Spring 2018

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Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 8.1.
- *Integral Calculus*, Khan Academy: Area & arc length using calculus.

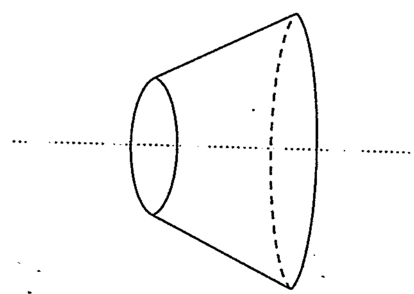
Keywords: surface area, surface of revolution

Applications of integration: Surface Area & Surfaces of Revolution

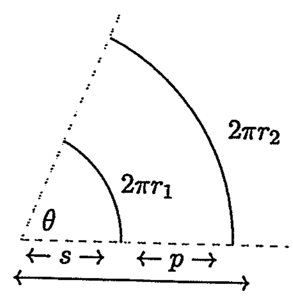
In the last lecture I was **frustrated by the frustrum!** Let's sort things out.

We are trying to determine the surface area of the general frustrum F obtained by rotating a line segment $y = mx + c$, $a \leq x \leq b$, around the x-axis.

$$y = mx + c$$
$$a \leq x \leq b$$



To determine the surface area of F we cut the frustrum along a line and unroll it in the plane we obtain a circular sector having radius r and angle θ with a concentric sector of radius $r - s$ removed.



We have

$$r_1 = ma + c, \qquad r_2 = mb + c$$

and using the formula for the length of the line segment $y = mx + c$, $x_1 \leq x \leq x_2$,

$$(x_2 - x_1)\sqrt{1 + m^2}$$

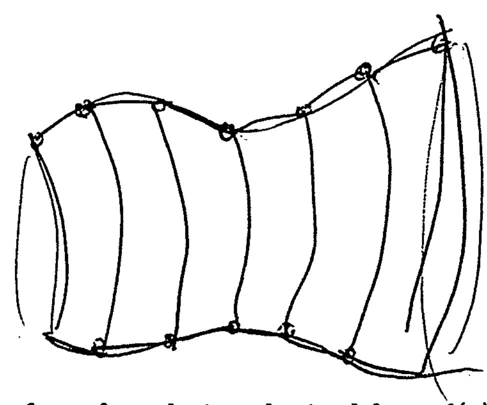
we determine
$$s = \left(a + \frac{c}{m}\right)\sqrt{1+m^2}, \qquad r = \left(b + \frac{c}{m}\right)\sqrt{1+m^2}$$
Moreover, since $p = r - s$, we obtain
$$\theta p = \theta(r-s) = 2\pi(r_2 - r_1) = 2\pi m(b-a)$$
Here we use the fact that $\theta s = 2\pi r_1$ (resp. $\theta r = 2\pi r_2$) is the length of a circular arc appearing above. Now, the area of F is given by
$$A = \frac{\theta}{2\pi}\pi(r^2 - s^2) = \frac{\theta}{2}p(r+s) = \pi m(b-a)\sqrt{1+m^2}\left(a+b+\frac{2c}{m}\right)$$
$$\implies A = 2\pi\sqrt{1+m^2}\left(\frac{m}{2}(b^2-a^2) + c(b-a)\right) \qquad (*)$$

PHEW!

CHECK YOUR UNDERSTANDING
Let $f(x) = mx + c$. Show that A given in $(*)$ can be computed using a definite integral:
$$A = 2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}dx$$
$$= 2\pi\int_a^b (mx+c)\sqrt{1+m^2}\,dx$$
$$= 2\pi\left(\sqrt{1+m^2}\right)\left[\frac{mx^2}{2} + cx\right]_a^b$$
$$= 2\pi\sqrt{1+m^2}\left[\frac{m(b^2-a^2)}{2} + c(b-a)\right].$$

We can now approximate the surface area of a surface of revolution using a collection of circular frustrums.

PICTURE



Let S be a surface of revolution obtained from $f(x)$, $a \le x \le b$. Choose a natural number n.

1. Subdivide $[a,b]$ into n subintervals having equal length so that the endpoints of each subinterval are
$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$

2. Define the piecewise linear function $g_n(x)$ as follows (it's the function whose graph is the collection of line segments above):

$$g_n(x) = m_i(x - x_i) + f(x_i), \qquad \text{when } x_{i-1} \leq x \leq x_i.$$

Here

$$m_i = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

The piecewise linear function $g_n(x)$ provides an approximation to the graph of $f(x)$

3. Then, the surface of revolution S is approximated by a collection of n circular frustrums $F_1, \ldots, F_n$ as in the above diagram. Moreover,

$$\text{Surface area of } F_i = 2\pi \int_{x_{i-1}}^{x_i} g_n(x)\sqrt{1 + g_n'(x)^2}\,dx$$

4. Hence, the surface area of S is obtained as the limit

$$\text{Surface area of } S = \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi \int_{x_{i-1}}^{x_i} \sqrt{1 + (g_n'(x))^2}\, g_n(x)\,dx$$

$$= 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\,dx.$$

Example: The surface area A of the surface of revolution about the x-axis obtained from $f(x) = 2\sqrt{x}$ when $1 \leq x \leq 2$ is

$$A = 2\pi \int_1^2 \sqrt{1 + \frac{1}{x}}\, 2\sqrt{x}\,dx = 4\pi \int_1^2 \sqrt{x+1}\,dx = 4\pi \left[\frac{2}{3}(x+1)^{3/2}\right]_1^2 = \frac{8\pi}{3}\left(\sqrt{27} - \sqrt{8}\right)$$

CHECK YOUR UNDERSTANDING

The surface of the ball of radius $a > 0$ can be realised as a surface of revolution of the function $f(x) = \sqrt{a^2 - x^2}$, $-a \le x \le a$.

1. Show that

$$1 + f'(x)^2 = \frac{a^2}{a^2 - x^2}$$

$$1 + f'(x)^2 = 1 + \left(\frac{-x}{\sqrt{a^2-x^2}}\right)^2$$

$$= 1 + \frac{x^2}{a^2-x^2}$$

$$= \frac{a^2 - x^2 + x^2}{a^2-x^2} = \frac{a^2}{a^2-x^2}$$

2. Show that

$$f(x)\sqrt{1 + f'(x)^2} = a$$

$$f(x)\sqrt{1+f'(x)^2} = \sqrt{\cancel{a^2-x^2}} \cdot \sqrt{\frac{a^2}{\cancel{a^2-x^2}}}$$

$$= a$$

3. Use the formula for surface area of a surface of revolution and deduce the well-known formula for the surface area A of a ball of radius a:

$$A = 4\pi a^2$$

$$\cancel{2\pi \int_{-a}^{a} \sqrt{a^2-x^2}\sqrt{1+}}$$

$$2\pi \int_{-a}^{a} f(x)\sqrt{1+f'(x)^2}\,dx = 2\pi \int_{-a}^{a} a\,dx$$

$$= 2\pi a \cdot 2a = \underline{\underline{4\pi a^2}}$$