Contact: `gmelvin@middlebury.edu`

April 2 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 11.8, 11.9
- *Power Series*, Integral Calculus, Khan Academy

Keywords: power series, interval of convergence

Power Series

Recall the series

$$1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

This series is convergent *for any* x and we are able to define the **exponential function**

$$\exp(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

with domain being the collection of all real numbers.

Check your understanding

Let x be a real number. Consider the geometric series

$$G(x) = 1 + \sum_{n=1}^{\infty} x^n$$

1. $G(x)$ converges for all x satisfying

$$\_\_\_\_\_\_\_\_ < x < \_\_\_\_\_\_\_\_$$

2. $G(x)$ defines a function with domain

$$\_\_\_\_\_\_\_\_ < x < \_\_\_\_\_\_\_\_$$

3. Since $\sum_{n=1}^{\infty} x^n =$ ____________, whenever this series converges,

$$G(x) = 1 + \sum_{n=1}^{\infty} x^n = \_\_\_\_\_\_\_\_\_\_\_\_$$

   whenever ________ $< x <$ ________

Hence, $G(x)$ gives a **series representation** of a well-known function ____________ whenever ________ $< x <$ ________.

Example:

1. The series

$$F(x) = 1 + \sum_{n=1}^{\infty} 2^n x^n$$

is convergent whenever

$$|2x| < \rule{4em}{0.4pt} \qquad \iff \qquad \rule{4em}{0.4pt} < x < \rule{4em}{0.4pt}$$

Since $F(x) = G(2x)$, we obtain

$$G(x) = 1 + \sum_{n=1}^{\infty} 2^n x^n = \rule{6em}{0.4pt}$$

whenever $\rule{4em}{0.4pt} < x < \rule{4em}{0.4pt}$

2. The series

$$H(x) = \frac{3}{2} + \sum_{n=1}^{\infty} 3\frac{x^n}{2^{n+1}}$$

is convergent whenever

$$\left|\frac{x}{2}\right| < \rule{4em}{0.4pt} \qquad \iff \qquad \rule{4em}{0.4pt} < x < \rule{4em}{0.4pt}$$

Since $H(x) = \frac{3}{2}F\left(\frac{x}{2}\right)$, we obtain

$$H(x) = \frac{3}{2}\left(1 + \sum_{n=1}^{\infty} \frac{x^n}{2^n}\right) = \rule{6em}{0.4pt} = \rule{6em}{0.4pt}$$

whenever $\rule{4em}{0.4pt} < x < \rule{4em}{0.4pt}$.

For the next couple of weeks we are going to investigate functions that can be represented by series, simialr to what we've seen above.

Definition: A **power series** is a series of the form

$$c_0 + \sum_{n \geq 1} c_n (x-c)^n$$

where $c_0, c_1, c_2, \ldots$ and c are constant, and x is a variable. We call c the **centre** of the power series, $c_0, c_1, \ldots$ the **coefficients** of the power series.

Remark:

1. Observe that a power series is completely determined by its centre c and the coefficients $c_0, c_1, c_2, c_3, \ldots$: any two power series possessing the same centre and coefficients are the same power series.

2. We will frequently write

$$\sum_{n=0}^{\infty} c_n (x-c)^n$$

as shorthand for a power series.

Example: The power series

$$1 + \sum_{n=1}^{\infty} \frac{(x+1)^n}{3^n(n+1)}$$

has centre $c = -1$ and $c_n = \frac{1}{3^n(n+1)^n}$, $n = 0, 1, 2, 3, \ldots$.

> **Basic Question:** For which x does a power series give a well-defined function?

Let's use the Ratio Test to determine where the series above is convergent. Let $a_n = \frac{(x+1)^n}{3^n(n+1)}$. Then,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+1)^{n+1}}{3^{n+1}(n+2)} \frac{3^n(n+1)}{(x+1)^n}\right| = |x+1| \left(\frac{1}{3}.\frac{n+1}{n+2}\right)$$

and,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |x+1| \left(\frac{1}{3}.\frac{n+1}{n+2}\right) = \frac{|x+1|}{3}$$

Hence, this series

- converges whenever $|x+1| <$ __________ $\implies$ __________ $< x <$ __________
- diverges whenever $|x+1| >$ __________ $\implies$ _________________________

What about when $|x+1| =$ __________? We have to check directly:

CHECK YOUR UNDERSTANDING

- $(x+1) =$ __________

- $(x+1) =$ __________

Hence, the series

$$1 + \sum_{n=1}^{\infty} \frac{(x+1)^n}{3^n(n+1)}$$

is convergent when ____________________ and divergent otherwise.

Definition: Let $\sum_{n=0}^{\infty} c_n(x-c)^n$ be a power series. The largest interval on which the power series converges is called the **interval of convergence**.

There are three possibilities for the interval of convergence of a power series:

1. the interval of convergence is a single point $x = c$;
2. the interval of convergence is a finite interval of the form

   $(c-R, c+R)$, or $[c-R, c+R]$, or $(c-R, c+R]$, or $[c-R, c+R)$

   for some R (the **radius of convergence**)
3. the interval of convergence is $(-\infty, \infty)$

Example:

1. Consider the exponential series

$$\exp(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

   This is a power series centred at $c = 0$, and $c_n = \frac{1}{n!}$, for $n = 0, 1, 2, 3, 4, \ldots$. The radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \right| = \lim_{n\to\infty} (n+1) = +\infty$$

   Hence, we recover the fact that $\exp(x)$ is well-defined for all x i.e. the series converges for all x.

2. Consider the power series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$$

   This is a power series centred at $c = 1$ and $c_n = \frac{1}{n}$, for $n = 1, 2, 3, \ldots$. The radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \frac{n+1}{n} = 1$$

   Hence, the power series

   (a) converges when $|x-1| < 1$ i.e. when $0 < x < 2$, and
   (b) diverges when $|x-1| > 1$ i.e. when $x > 2$ or $x < 0$.

   If $|x-1| = 1$ then $x = 0$ or $x = 2$ and we have two separate cases to consider:

   - $x = 0$: In this case the power series is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

This series is convergent by the Alternating Series Test.

- $x = 2$: In this case the power series is

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

This series is divergent.

Hence, the power series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$ is convergent when $0 \leq x < 2$, and divergent otherwise.

3. Consider the power series

$$\sum_{n=0}^{\infty} n!(1-x)^n = \sum_{n=0}^{\infty} n!(-1)^n(x-1)^n$$

We have coefficients $c_n = (-1)^n n!$. The centre of the power series is $c = 1$ and the radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^n n!}{(-1)^{n+1}(n+1)!} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Hence, the radius of convergence is $R = 0$. Thus, the series converges at $x = 1$ and diverges for $x \neq 1$.