Middlebury College

Calculus II: Spring 2018

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April 26 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 8.1.
- - *Integral Calculus*, Khan Academy: Area & arc length using calculus.

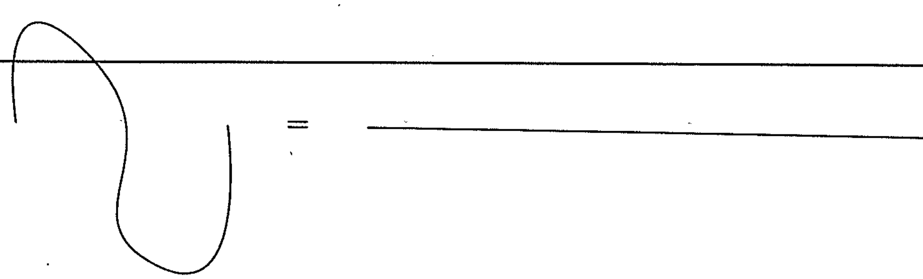
Keywords: arc length

Applications of Integration: Arc Length

In this lecture we will investigate the notion of *arc length*. We define what we mean by the *length* of the graph of a continuous function $f(x)$, and deduce an integration formula for the arc length of a large class of curves in the plane.

Lines and length

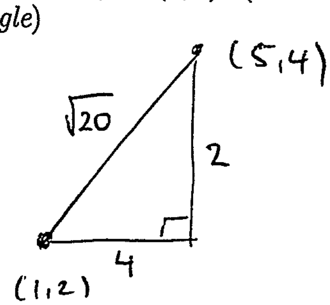
Given a curve C in the plane, the notion of what we mean by its length seems intuitive: take a piece of string, or some other pliable material, lay the string along the curve and mark the enbpoints. After pulling the string taut we can measure the length of the curve - ta-da!



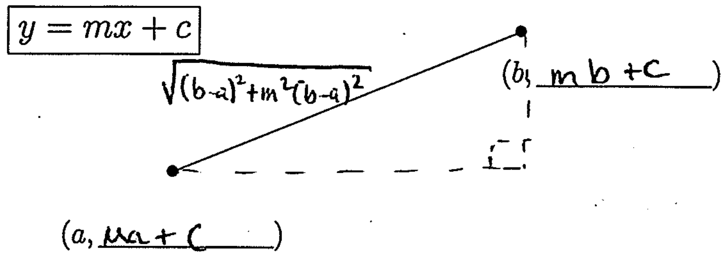
But how do we perform this procedure **in practice?** Let's start out simple and look at straight line segments.

Mathematical workout - flex those muscles!

1. Find the length of the straight line segment from $(1,2)$ to $(5,4)$. (*Hint: realise the straight line segment as the hypotenuse of a right triangle*)

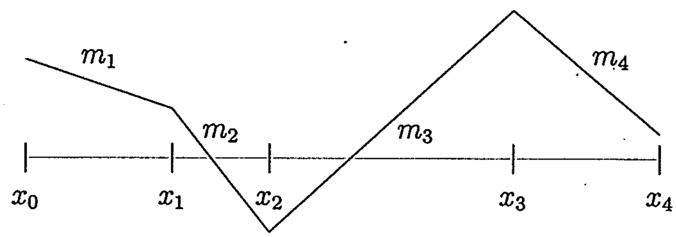


2. Generalise the above example to find a formula for the length of the segment of the straight line $y = mx + c$ from $x = a$ to $x = b$.



Length $= \underline{(b-a)\sqrt{1+m^2}}$

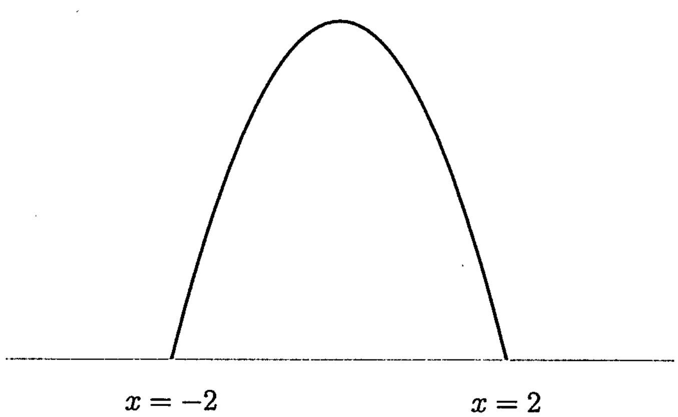
3. What's the length of the following curve? Each straight line segment has slope m_1, m_2, m_3, m_4.



$$(x_1 - x_0)\sqrt{1+m_1^2} + (x_2 - x_1)\sqrt{1+m_2^2} + (x_3 - x_2)\sqrt{1+m_3^2} + (x_4 - x_3)\sqrt{1+m_4^2}$$

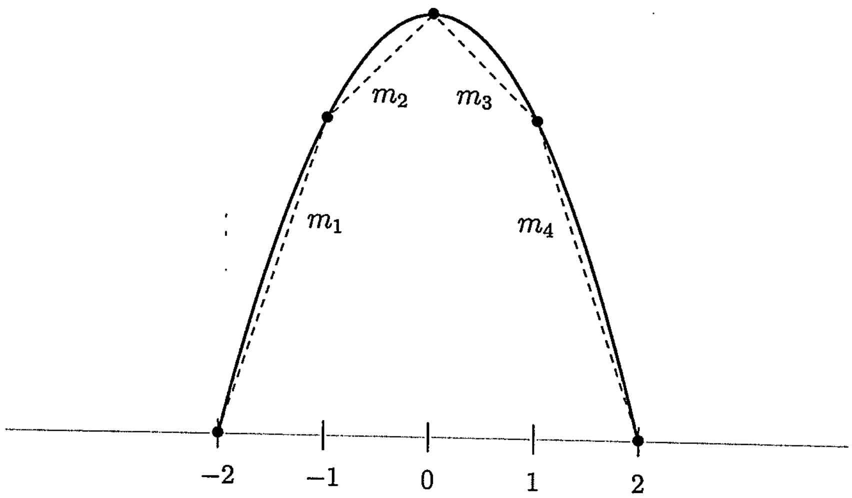
The arc length formula

Consider the function $f(x) = 4 - x^2$. How might we determine the length of the parabolic curve segment between $x = -2$ and $x = 2$?



Check your understanding

Subdivide the interval $[-2, 2]$ into four segments of equal length as demonstrated below



1. Write down the slopes m_1, m_2, m_3 and m_4 of the line segments drawn above

$$\boxed{m_1 = \underline{\ 3\ } \qquad m_2 = \underline{\ 1\ } \qquad m_3 = \underline{\ -1\ } \qquad m_4 = \underline{\ -3\ }}$$

2. Determine the total length of curve consisting of the ~~three~~ four dashed straight line segments above. How is this distance related to the actual length of the parabolic curve segment?

$$\sqrt{1+3^2} + \sqrt{1+1^2} + \sqrt{1+(-3)^2} + \sqrt{1+(-1)^2}$$

$$= 2\left(\sqrt{10} + \sqrt{2}\right)$$

3. Determine x_1, x_2, x_3, x_4 in the interval $[-2, 2]$ such that

$$f'(x_1) = m_1, \qquad f'(x_2) = m_2, \qquad f'(x_3) = m_3, \qquad f'(x_4) = m_4.$$

Solve $\quad -2x = m_i \,, \quad$ for $i = 1, 2, 3, 4$

$\Rightarrow \quad x_1 = -3/2 \qquad x_2 = -1/2 \qquad x_3 = 1/2 \qquad x_4 = 3/2$

4. How can you interpret the previous problem using the graph of $f(x)$?

$f'(x_i) =$ slope of tangent line to $y = f(x)$ at $(x_i, f(x_i))$

$\Rightarrow$ tangent line parallel to line segment drawn above.

Let's consider what happens if we subdivide the interval $[-2,2]$ into n segments having equal length and look to *approximate* the length L of the parabolic curve segment using a piecewise linear curve C_n (i.e. a collection of n straight line segments, analogous to what we did above).

1. First we observe that each segment will have length $\underline{4/n}$.
2. If each straight line segment in C_n has slope $m_1, m_2, \ldots, m_n$, respectively, then

$$\text{Length of } C_n = \sum_{i=1}^{n} \frac{4}{n}\sqrt{1+m_i^2}$$

3. In the j^{th} segment of $[-2,2]$, we can find x_j such that $f'(x_j) = \underline{m_j}$. Hence, we can rewrite

$$\text{Length of } C_n = \sum_{i=1}^{n} \frac{4}{n}\sqrt{1+f'(x_j)^2}$$

Now, as n gets very large, we expect that the length of C_n will give a good approximation of the length of the parabolic curve segment from $x=-2$ to $x=2$. It seems reasonable, therefore, that we should be able to write

$$L = \lim_{n\to\infty} \text{Length of } C_n = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\sqrt{1+f'(x_j)^2}$$

We recognise this last expression as the Riemann sum associated to the function

$$g(x) = \sqrt{1+f'(x)^2}$$

Let $f(x)$ be a function defined on the interval $[a,b]$ for which the derivative $f'(x)$ is continuous (this will be the case for all functions we encounter). Then, we define the **arc length of the curve** $y = f(x)$ **between** $x = a$ **to** $x = b$ to be

$$\int_a^b \sqrt{1+f'(x)^2}\, dx$$

Remark: The condition that $f'(x)$ is continuous is required to ensure that the integral (i.e. a limit of Riemann sums) is well-defined.

In order to determine definite integrals, we need to recall the following essential result.

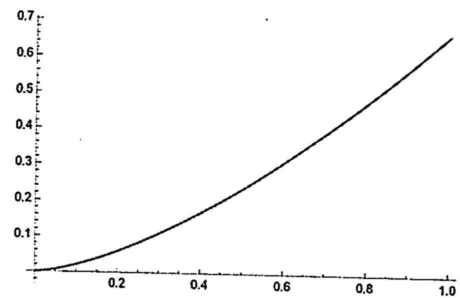
Fundamental Theorem of Calculus, Revisited
Let $F(x)$ be an antiderivative of $f(x)$. Then,

$$\int_a^b f(x)dx = F(b) - F(a)$$

Example:

1. The arc length of the curve $y = \frac{2}{3}x^{\frac{3}{2}}$ between $x = 0$ and $x = 1$ is given by

$$\int_0^1 \sqrt{1+x}dx \overset{u=1+x}{=} \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^1 = \frac{2\sqrt{8}}{3} - \frac{2}{3}$$



2. The arc length of the curve $y = 4 - x^2$ between $x = -2$ and $x = 2$ is given by

$$\int_{-2}^2 \sqrt{1+4x^2}dx.$$

To determine this integral we will need to make an inverse trigonometric substitution $x = \frac{1}{2}\tan(t)$. This leads to the definite integral

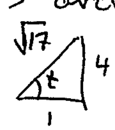
$$2\int_0^{\overset{\arctan(4)}{\cancel{\arctan(2)}}} \sec^3(t)dt$$

When $x = 2$

$2 = \frac{1}{2}\tan(t)$

$\Rightarrow 4 = \tan(t)$

$\Rightarrow \arctan(4) = t$



Using integration by parts

$$\begin{aligned} f &= \sec(t) & g' &= \sec^2(t) \\ f' &= \sec(t)\tan(t) & g &= \underline{\tan(t)} \end{aligned}$$

and

$$\int \sec^3(t)dt = \sec(t)\tan(t) - \int \sec(t)\tan^2(t)dt = \sec(t)\tan(t) + \underline{\int \sec(t)\,dt - \int \sec^3(t)dt}$$

$$\Longrightarrow \int \sec^3(t)dt = \underline{\frac{1}{2}\left[\sec(t)\tan(t) + \ln|\sec(t) + \tan(t)|\right]}$$

$$\Rightarrow 2\int_0^{\arctan(4)} \sec^3(t)dt = 2\sqrt{17} + \ln\left(\sqrt{17}+4\right).$$

3. Let $f(x) = \frac{x^3}{6} + \frac{1}{2x}$, so that $f'(x) = \frac{x^2}{2} - \frac{1}{2x^2}$. The arc length of the curve $y = f(x)$ between $x = 1$ and $x = 3$ is given by

$$\begin{aligned}
\int_1^3 \sqrt{1 + \frac{1}{4}(x^2 - x^{-2})^2 dx} &= \int_1^3 \sqrt{x^2 + \frac{1}{2} + x^{-2} dx} \\
&= \int_1^3 \sqrt{\frac{1}{4}(x^2 + x^{-2})^2 dx} \\
&= \frac{1}{2}\int_1^3 (x^2 + x^{-2}) dx, \quad \text{since } x^2 + x^{-2} \geq 0 \text{ whenever } 1 \leq x \leq 3, \\
&= \frac{1}{2}\left[\frac{x^3}{3} - \frac{1}{x}\right]_1^3 = \frac{14}{3}
\end{aligned}$$

CHECK YOUR UNDERSTANDING

Determine the arc length of $y = 2(x-1)^{3/2}$ between $x = 1$ and $x = 5$.

$$f(x) = 2(x-1)^{3/2}$$

$$f'(x) = 3(x-1)^{1/2}$$

$$\int_1^5 \sqrt{1 + f'(x)^2}\, dx = \int_1^5 \sqrt{1 + 9(x-1)}\, dx$$

$$= \int_1^5 \sqrt{9x - 8}\, dx$$

$$= \left[\frac{2}{27}(9x-8)^{3/2}\right]_1^5$$

$$= \frac{2}{27}\left(\sqrt{37^3} - 1\right)$$