MATH 122 : HW 4/23

1a) ~~x = a sin(t)~~ $\qquad t = a\sin(u)$

$\cancel{\frac{dx}{dt} = a\cos(t)} \qquad \frac{dt}{du} = a\cos(u)$

$$\int_0^x \sqrt{a^2-t^2}\,dt = \int_{u=0}^{u=\arcsin(\frac{x}{a})} \sqrt{a^2-a^2\sin^2(u)}\cdot a\cos(u)\,du$$

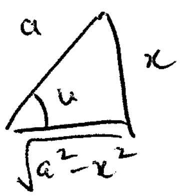
$$= a^2 \int_{u=0}^{u=\arcsin(x/a)} \cos^2(u)\,du$$

$$= a^2 \int_0^{\arcsin(x/a)} \frac{1}{2}\left(1+\cos(2u)\right)du$$

$u = \arcsin\left(\frac{x}{a}\right)$

$\sin(u) = \frac{x}{a}$

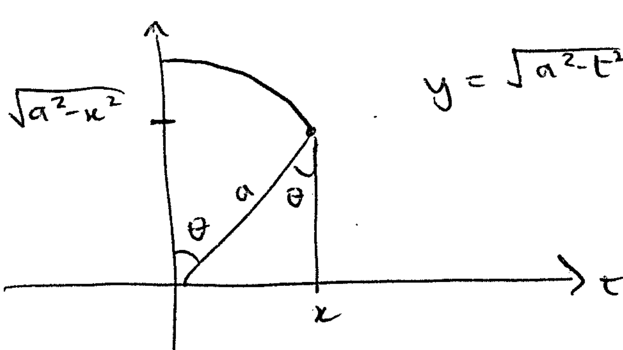
$$= \left[\frac{a^2}{2}\left(u + \sin(u)\cos(u)\right)\right]_0^{\arcsin(\frac{x}{a})}$$



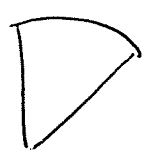
$$= \frac{a^2}{2}\left(\arcsin\left(\frac{x}{a}\right) + \frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}\right)$$

$$= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2-x^2}.$$

b)



$y = \sqrt{a^2-t^2}$

$$\int_0^x \sqrt{a^2-t^2}\,dt = \text{area below arc from } t=0 \text{ to } t=x. \; = $$  + 

- The triangle has area

$$\frac{1}{2} x . \sqrt{a^2 - x^2}$$

- The wedge has area

$$\frac{\theta}{2\pi} . \pi a^2$$

$$= \frac{\theta}{2} a^2$$

and $\sin\theta = \frac{x}{a}$

$\Rightarrow \theta = \arcsin\left(\frac{x}{a}\right)$

ie wedge has area

$$\frac{a^2}{2} \arcsin\left(\frac{x}{a}\right).$$

2a) $\frac{5x+1}{(x-1)(2x+1)} = \frac{A}{x-1} + \frac{B}{2x+1}$

$\Rightarrow 5x+1 = A(2x+1) + B(x-1)$

Input	LHS		RHS	
$x = 1$	6	$=$	$3A$	$\Rightarrow A = 2$
$x = -1/2$	$-3/2$	$=$	$-\frac{3}{2}B$	$\Rightarrow B = 1$

Hence, $\frac{5x+1}{(2x+1)(x-1)} = \frac{2}{x-1} + \frac{1}{2x+1}.$

2b) $$\frac{2}{x^2+3x+2} = \frac{2}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$$

$$\Rightarrow \quad 2 = A(x+1) + B(x+2)$$

Input	LHS		RHS	
$x=-1$	2	$=$	B	$B=2$
$x=-2$	2	$=$	$-A$	$A=-2$

$$\Rightarrow \quad \frac{2}{x^2+3x+2} = \frac{-2}{x+2} + \frac{2}{x+1}$$

2c) $$\frac{3x-4}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$\Rightarrow \quad 3x-4 = A(x+2) + B(x-3)$$

Input	LHS		RHS	
$x=-2$	-10	$=$	$-5B$	$\Rightarrow B=2$
$x=3$	5	$=$	$5A$	$\Rightarrow A=1$

$$\Rightarrow \quad \frac{3x-4}{(x-3)(x+2)} = \frac{1}{x-3} + \frac{2}{x+2}.$$

d) $\dfrac{2x^2}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

$\Rightarrow \quad 2x^2 = A(x+1)^2 + Bx(x+1) + Cx$

Input	LHS	RHS
$x=0$	0	$= A$
$x=-1$	2	$= -C$
$x=1$	2	$= 2B + C$

$\Rightarrow \quad A=0 \quad B=2 \quad C=-2$

$\dfrac{2x^2}{x(x+1)^2} = \dfrac{2x}{(x+1)^2} = \dfrac{2}{x+1} - \dfrac{2}{(x+1)^2}$

e) $\dfrac{x+2}{(x^2-1)^2} = \dfrac{x+2}{(x-1)^2(x+1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2}$

$\Rightarrow \quad x+2 = A(x-1)(x+1)^2 + B(x+1)^2 + C(x-1)^2(x+1) + D(x-1)^2$

Input	LHS	RHS	
$x=1$	3	$= 4B$	$\Rightarrow B = 3/4$
$x=-1$	-1	$= 4D$	$\Rightarrow D = -1/4$
$x=0$	2	$= -A + \frac{3}{4} + C - 1/4$	
$x=2$	4	$= 9A + \frac{3}{4}\cdot 9 + 3C - 1/4$	

$$\frac{3}{2} = -A + C \quad (1)$$

$$-3 = 9A + 3C \quad (2)$$

$$9 \times (1) + (2) : \frac{21}{2} = 12C \Rightarrow C = \frac{21}{24} = \frac{7}{8}$$

$$(1) \Rightarrow \frac{3}{2} = -A + \frac{7}{8}$$

$$\Rightarrow A = -\frac{5}{8}$$

$$\Rightarrow \frac{x+2}{(x^2-1)^2} = -\frac{5}{8}\frac{1}{x-1} + \frac{3}{4}\frac{1}{(x-1)^2} + \frac{7}{8}\cdot\frac{1}{x+1} - \frac{1}{4}\frac{1}{(x+1)^2}$$

f) $$\frac{x+b}{x^2-a^2} = \frac{x+b}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$$

$$x+b = A(x+a) + B(x-a)$$

Input	LHS	RHS
$x = a$	$a+b = 2aA$	$\Rightarrow A = \frac{1}{2}\left(1+\frac{b}{a}\right)$
$x = -a$	$-a+b = -2aB$	$\Rightarrow B = \frac{1}{2}\left(1-\frac{b}{a}\right)$

$$\Rightarrow \frac{x+b}{x^2-a^2} = \frac{1}{2}\frac{1+b/a}{x-a} + \frac{1}{2}\frac{1-b/a}{x+a}.$$