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April 19 Lecture

Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 7.4.
- *Calculus*, Spivak, 3rd Ed.: Section 19.
- *AP Calculus BC*, Khan Academy: Antiderivatives and the fundamental theorem of calculus.

Keywords: partial fractions

Techniques of Integration IV. Partial Fractions.

Partial fractions: the case of distinct linear factors

The method of partial fractions was introduced yesterday can be extended to the case when the denominator of a rational function can be factored as a product of **distinct linear factors.**

Example: Determine

$$\int \frac{2x^2+1}{(x^2-1)(x^2-4)}dx$$

As the degree of the numerator is less than the degree of the denominator we do not need to perform long division. Observe that

$$(x^2-1)(x^2-4) = \underline{(x-1)(x+1)(x-2)(x+2)}$$

is a product of *distinct linear factors.* Based on our investigations yesterday, we might expect to be able to write

$$\frac{2x^2+1}{(x^2-1)(x^2-4)} = \frac{2x^2+1}{(x-1)(x+1)(x-2)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2} + \frac{D}{x+2}$$

Rearranging and taking acommon denominator leads to the equality of numerators

$$2x^2+1 = A\underline{(x+1)(x^2-4)} + B\underline{(x-1)(x^2-4)} + C\underline{(x^2-1)(x+2)} + D\underline{(x^2-1)(x-2)}$$

This last equality is an *equality of functions*: inputting a value for x on either side will give the same output. In particular,

Input $x=1$: $2.1^2+1 = \underline{-6A}$ $\Longrightarrow A = \underline{-1/2}$

Input $x=-1$: $2.(-1)^2+1 = \underline{6B}$ $\Longrightarrow B = \underline{1/2}$

Input $x=2$: $2.2^2+1 = \underline{12C}$ $\Longrightarrow C = \underline{3/4}$

Input $x=-2$: $2.(-2)^2+1 = \underline{-12D}$ $\Longrightarrow D = \underline{-3/4}$

Hence,

$$\int \frac{2x^2+1}{(x^2-1)(x^2-4)}dx = \int \frac{-\frac{1}{2}}{x-1} + \frac{1}{2}\frac{1}{x+1} + \frac{3}{4}\frac{1}{x-2} - \frac{3}{4}\frac{1}{x+2}\,dx$$

$$= -\frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + \frac{3}{4}\ln|x-2| - \frac{3}{4}\ln|x+2| + C$$

CHECK YOUR UNDERSTANDING

Determine

$$\int \frac{2x^2+x+1}{x^3-4x}dx$$

$$\frac{2x^2+x+1}{x^3-4x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

$$\Rightarrow \quad 2x^2+x+1 = A(x-2)(x+2) + Bx(x+2) + Cx(x-2)$$

Input	LHS		RHS	
$x=0$	1	$=$	$-4A$	$\Rightarrow A = -1/4$
$x=2$	11	$=$	$8B$	$\Rightarrow B = 11/8$
$x=-2$	7	$=$	$-8B$	$\Rightarrow C = -7/8$

$$\int \frac{2x^2+x+1}{x^3-4x}\,dx = \int -\frac{1}{4x} + \frac{11}{8(x-2)} - \frac{7}{8(x+2)}\,dx$$

$$= -\frac{1}{4}\ln|x| + \frac{11}{8}\ln|x-2| - \frac{7}{8}\ln|x+2| + C.$$

Method of Partial Fractions Case I: distinct linear factors

Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function, where

$$Q(x) = (a_1x+a_2)(b_1x+b_2)\cdots(p_1x+p_2)$$

is a product of *distinct* linear factors (i.e. none of the factors repeat and no factor is a constant multiple of another).

- If $\deg P(x) \geq \deg Q(x)$ perform long division and write

$$\frac{P(x)}{Q(x)} = b(x) + \frac{r(x)}{Q(x)}$$

Proceed to the next step, replacing $P(x)$ by $r(x)$.

- If $\deg P(x) < \deg Q(x)$ determine constants $A, B, \ldots, P$ so that

$$\frac{P(x)}{Q(x)} = \frac{A}{a_1x+a_2} + \frac{B}{b_1x+b_2} + \ldots + \frac{P}{p_1x+p_2}$$

It is a fact that the constants $A, B, \ldots, P$ you determine are **unique.**

Partial fractions: repeated linear factors

In this paragraph we will extend the method of partial fractions to the case of **repeated linear factors**. The problem here is that if we try to mimic the approach above, there are several possible routes to take.

Example: Consider the antiderivative problem

$$f(x) = \frac{2x-1}{x(x-1)^2}$$

FACT: there exist unique constants A, B, C so that

$$\frac{2x-1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

MATHEMATICAL WORKOUT

1. Determine constants A, B, C so that

$$A(x-1)^2 + Bx(x-1) + Cx = 2x - 1$$

Input	LHS		RHS
$x=0$	A	$=$	-1
$x=1$	C	$=$	1
$x=-1$	$(-1)(-2)^2 + (-1)\cdot(-2)B + (-1)$	$=$	$-3 \Rightarrow B = 1$

2. Complete the following antiderivative problem

$$\int \frac{2x-1}{x(x-1)^2}dx = \int \frac{-1}{x} + \frac{1}{x-1} + \frac{1}{(x-1)^2} \, dx$$

$$= -\ln|x| + \ln|x-1| - \frac{1}{x-1} + C.$$

SPOT THE PATTERN!

Make an educated guess to complete the following statements:

1. There exists unique constants A, B, C so that

$$\frac{x^2}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

2. There exists unique constants A, B, C, D so that

$$\frac{x^3+1}{x^2(x+2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+2)} + \frac{D}{(x+2)^2}$$

3. There exists unique constants A, B, C, D, E, F so that

$$\frac{x+1}{(x^2-4)(x-1)^2x^2} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{x} + \frac{F}{x^2}$$

The above decompositions of a rational function is known as a **partial fraction decomposition**.

MATHEMATICAL WORKOUT

Determine constants A, B, C so that

$$A(x-1)^2 + B(x-1) + C = x^2$$

$$\frac{x^2}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

Deduce

$$\int \frac{x^2}{(x-1)^3}dx$$

Input	LHS		RHS
$x=1$	C	$=$	1
$x=0$	$A - B + 1$	$=$	0
$x=2$	$A + B + 1$	$=$	4

$$A - B = -1$$
$$A + B = 3$$
$$2A + 0 = 2$$
$$\Rightarrow A = 1$$
$$\Rightarrow B = 2$$

$$\int \frac{x^2}{(x-1)^3}dx = \int \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{1}{(x-1)^3}\,dx$$

$$= \ln|x-1| - \frac{2}{x-1} - \frac{1}{2}\frac{1}{(x-1)^2} + C.$$

Method of Partial Fractions Case II: repeated linear factors

Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function, where

$$Q(x) = (a_1x + a_2)^r(b_1x + b_2)^s \cdots (c_1x + c_2)^t$$

is a product of (possibly repeated) linear factors.

- If $\deg P(x) \geq \deg Q(x)$ perform long division and write

$$\frac{P(x)}{Q(x)} = b(x) + \frac{r(x)}{Q(x)}$$

Proceed to the next step, replacing $P(x)$ by $r(x)$.

- If $\deg P(x) < \deg Q(x)$ determine constants $A_1, \ldots, A_r, B_1, \ldots, B_s, \ldots, C_1, \ldots, C_t$ so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + a_2} + \frac{A_2}{(a_1x + a_2)^2} + \cdots + \frac{A_r}{(a_1x + a_2)^r} + \frac{B_1}{b_1x + b_2} + \cdots + \frac{B_s}{(b_1x + b_2)^s} + \cdots$$

It is a fact that the constants $A_1, \ldots, A_r, B_1, \ldots, B_s, \ldots, C_1, \ldots, C_t$ you determine are unique.

CHECK YOUR UNDERSTANDING

Determine

$$\int \frac{1 - 2x^2}{x^4 + 2x^3 + x^2} dx$$

by completing the following steps.

1. Factorise $x^4 + 2x^3 + x^2$ as a product of linear factors.

$$x^4 + 2x^3 + x^2 = x^2(x+1)^2$$

2. Determine a partial fraction decomposition

$$\frac{1 - 2x^2}{x^4 + 2x^3 + x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

$$1 - 2x^2 = Ax(x+1)^2 + B(x+1)^2 + Cx^2(x+1) + Dx^2$$

Input	LHS		RHS
$x=0$	$\boxed{1}$	$=$	$\boxed{B}$
$x=-1$	$\boxed{-1}$	$=$	$\boxed{D}$
$x=-2$	-7	$=$	$-2A + 1 - 4C - 4$
$x=1$	-1	$=$	$4A + 4 + 2C - 1$

$$\left.\begin{array}{l} -7 = -2A + 1 - 4C - 4 \\ -1 = 4A + 4 + 2C - 1 \end{array}\right\} \begin{array}{l} -4 = -2A - 4C \\ -4 = 4A + 2C \end{array}$$

Solving: $\boxed{C = 2}$, $\boxed{A = -2}$

3. Use the previous problems to obtain

$$\int \frac{1-2x^2}{x^4+2x^3+x^2}dx$$

$$= \int \frac{-2}{x} + \frac{1}{x^2} + \frac{2}{x+1} - \frac{1}{(x+1)^2} \, dx$$

$$= -2\ln|x| - \frac{1}{x} + 2\ln|x+1| + \frac{1}{x+1} + C$$

Exercise: Use the method of partial fractions to determine

$$\int \frac{x^4+x+1}{x^4-x^2}dx$$

Careful! You will need to perform long divison first.