Middlebury College

Calculus II: Spring 2018

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Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 7.4.
- *Calculus*, Spivak, 3rd Ed.: Section 19.
- *AP Calculus BC*, Khan Academy: Antiderivatives and the fundamental theorem of calculus.

Keywords: partial fractions

Techniques of Integration IV. Partial Fractions.

Partial fractions: the case of distinct linear factors

The method of partial fractions was introduced yesterday can be extended to the case when the denominator of a rational function can be factored as a product of **distinct linear factors**.

Example: Determine

$$\int \frac{2x^2+1}{(x^2-1)(x^2-4)}dx$$

As the degree of the numerator is less than the degree of the denominator we do not need to perform long division. Observe that

$$(x^2-1)(x^2-4) = \underline{\hspace{6cm}},$$

is a product of *distinct linear factors*. Based on our investigations yesterday, we might expect to be able to write

$$\frac{2x^2+1}{(x^2-1)(x^2-4)} = \frac{2x^2+1}{\underline{\hspace{5cm}}} = \frac{A}{\underline{\hspace{2cm}}} + \frac{B}{\underline{\hspace{2cm}}} + \frac{C}{\underline{\hspace{2cm}}} + \frac{D}{\underline{\hspace{2cm}}}$$

Rearranging and taking acommon denominator leads to the equality of numerators

$$2x^2+1 = A\underline{\hspace{3cm}} + B\underline{\hspace{3cm}} + C\underline{\hspace{3cm}} + D\underline{\hspace{3cm}}$$

This last equality is an *equality of functions*: inputting a value for x on either side will give the same output. In particular,

$$\begin{aligned}
\text{Input } x=1: &\quad 2.1^2+1 = \underline{\hspace{3cm}} \implies A = \underline{\hspace{2cm}}\\
\text{Input } x=-1: &\quad 2.(-1)^2+1 = \underline{\hspace{3cm}} \implies B = \underline{\hspace{2cm}}\\
\text{Input } x=2: &\quad 2.2^2+1 = \underline{\hspace{3cm}} \implies C = \underline{\hspace{2cm}}\\
\text{Input } x=-2: &\quad 2.(-2)^2+1 = \underline{\hspace{3cm}} \implies D = \underline{\hspace{2cm}}
\end{aligned}$$

Hence,

$$\int \frac{2x^2+1}{(x^2-1)(x^2-4)}dx = \int \underline{\hspace{8cm}} dx$$

$$= \underline{\hspace{9cm}}$$

CHECK YOUR UNDERSTANDING

Determine

$$\int \frac{2x^2+x+1}{x^3-4x}dx$$

Method of Partial Fractions Case I: distinct linear factors

Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function, where

$$Q(x) = (a_1x+a_2)(b_1x+b_2)\cdots(p_1x+p_2)$$

is a product of *distinct* linear factors (i.e. none of the factors repeat and no factor is a constant multiple of another).

- If $\deg P(x) \geq \deg Q(x)$ perform long division and write

$$\frac{P(x)}{Q(x)} = b(x) + \frac{r(x)}{Q(x)}$$

Proceed to the next step, replacing $P(x)$ by $r(x)$.

- If $\deg P(x) < \deg Q(x)$ determine constants $A, B, \ldots, P$ so that

$$\frac{P(x)}{Q(x)} = \frac{A}{a_1x+a_2} + \frac{B}{b_1x+b_2} + \ldots + \frac{P}{p_1x+p_2}$$

It is a fact that the constants $A, B, \ldots, P$ you determine are **unique**.

Partial fractions: repeated linear factors

In this paragraph we will extend the method of partial fractions to the case of **repeated linear factors**. The problem here is that if we try to mimic the approach above, there are several possible routes to take.

Example: Consider the antiderivative problem

$$f(x) = \frac{2x-1}{x(x-1)^2}$$

FACT: there exist unique constants A, B, C so that

$$\frac{2x-1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

MATHEMATICAL WORKOUT

1. Determine constants A, B, C so that

$$A(x-1)^2 + Bx(x-1) + Cx = 2x - 1$$

2. Complete the following antiderivative problem

$$\int \frac{2x-1}{x(x-1)^2}dx = \int \underline{\hspace{8cm}} dx$$

$$= \underline{\hspace{10cm}}$$

SPOT THE PATTERN!

Make an educated guess to complete the following statements:

1. There exists unique constants A, B, C so that

$$\frac{x^2}{(x-1)^3} = \frac{A}{\underline{\hspace{1.5cm}}} + \frac{B}{\underline{\hspace{1.5cm}}} + \frac{C}{\underline{\hspace{1.5cm}}}$$

2. There exists unique constants ______________ so that

$$\frac{x^3+1}{\qquad\qquad} = \frac{\qquad}{x} + \frac{\qquad}{x^2} + \frac{\qquad}{(x+2)} + \frac{\qquad}{(x+2)^2}$$

3. There exists unique constants _______________________ so that

$$\frac{x+1}{(x^2-4)(x-1)^2x^2} = \underline{\hspace{12cm}}$$

The above decompositions of a rational function is known as a **partial fraction decomposition**.

Mathematical workout

Determine constants A, B, C so that

$$A(x-1)^2 + B(x-1) + C = x^2$$

Deduce

$$\int \frac{x^2}{(x-1)^3} dx$$

Method of Partial Fractions Case II: repeated linear factors

Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function, where

$$Q(x) = (a_1x + a_2)^r(b_1x + b_2)^s \cdots (c_1x + c_2)^t$$

is a product of (possibly repeated) linear factors.

- If $\deg P(x) \geq \deg Q(x)$ perform long division and write

$$\frac{P(x)}{Q(x)} = b(x) + \frac{r(x)}{Q(x)}$$

Proceed to the next step, replacing $P(x)$ by $r(x)$.

- If $\deg P(x) < \deg Q(x)$ determine constants $A_1, \ldots, A_r, B_1, \ldots, B_s, \ldots, C_1, \ldots, C_t$ so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + a_2} + \frac{A_2}{(a_1x + a_2)^2} + \ldots + \frac{A_r}{(a_1x + a_2)^r} + \frac{B_1}{b_1x + b_2} + \ldots + \frac{B_s}{(b_1x + b_2)^s} + \ldots$$

It is a fact that the constants $A_1, \ldots, A_r, B_1, \ldots, B_s, \ldots, C_1, \ldots, C_t$ you determine are **unique**.

Check your understanding

Determine

$$\int \frac{1 - 2x^2}{x^4 + 2x^3 + x^2} dx$$

by completing the following steps.

1. Factorise $x^4 + 2x^3 + x^2$ as a product of linear factors.

2. Determine a partial fraction decomposition

$$\frac{1 - 2x^2}{x^4 + 2x^3 + x^2} = \underline{\hspace{6cm}}$$

3. Use the previous problems to obtain

$$\int \frac{1 - 2x^2}{x^4 + 2x^3 + x^2} dx$$

Exercise: Use the method of partial fractions to determine

$$\int \frac{x^4 + x + 1}{x^4 - x^2} dx$$

Careful! You will need to perform long divison first.