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April 18 Lecture

Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 7.4.
- *Calculus*, Spivak, 3rd Ed.: Section 19.
- *AP Calculus BC*, Khan Academy: Antiderivatives and the fundamental theorem of calculus.

Keywords: polynomial long division, partial fractions

Techniques of Integration IV. Partial Fractions.

In this lecture we will focus on the method of partial fractions. This method allows us to split up a rational function as a sum of elementary rational functions. In this way we will devise an approach to solving the antiderivative problem for a large class of rational functions.

Rational Functions

Let $f(x)$ be a function.

1. We say that $f(x)$ is a **polynomial function** if it can be written

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$

where $a_n, \ldots, a_0$ are constants. If $a_n \neq 0$ then we say that $f(x)$ **has degree** n and write $\deg f(x) = n$.

2. We say that $f(x)$ is a **rational function** if it can be written

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomial functions.

Example The following functions are polynomials having degree $3, 5, 8$

$$f(x) = 2x^3 - x^2 + 10x + 1, \qquad g(x) = 5x^5 + 3x - 1, \qquad h(x) = 7x^8 + 2x^2.$$

The following functions are rational functions

$$\frac{7x^8 + 2x^2}{2x^3 - x^2 + 10x + 1}, \qquad \frac{5x^5 + 3x - 1}{7x^8 + 2x^2}, \qquad \frac{5x^5 + 3x - 1}{2x^3 - x^2 + 10x + 1}$$

Long division of polynomial functions

Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function where $\deg P(x) \geq \deg Q(x)$. Then, there exists polynomial functions $b(x)$ and $r(x)$, where $\deg r(x) < \deg Q(x)$ so that

$$f(x) = \frac{P(x)}{Q(x)} = b(x) + \frac{r(x)}{Q(x)}$$

Example: Consider the rational function

$$f(x) = \frac{4x^4 - 6x^3 + x - 3}{2x^2 - x + 4}$$

Then, the method of long division of polynomial proceeds as follows (it's analogous to long division of integers - replace x by 10):

$$\begin{array}{r|rrrrr}
 & & 2x^2 & -2x & -5 & \\
\hline
2x^2 - x + 4 & 4x^4 - 6x^3 & & +x & -3 & \\
 & -4x^4 + 2x^3 & -8x^2 & & & \\
\cline{2-3}
 & -4x^3 & -8x^2 & +x & & \\
 & 4x^3 & -2x^2 & +8x & & \\
\cline{2-4}
 & & -10x^2 & +9x & -3 & \\
 & & 10x^2 & -5x & +20 & \\
\cline{3-5}
 & & & 4x & +17 &
\end{array}$$

Hence, $b(x) = 2x^2 - 2x - 5$ and $r(x) = 4x + 17$. You can check that

$$\frac{4x^4 - 6x^3 + x - 3}{2x^2 - x + 4} = 2x^2 - 2x - 5 + \frac{4x + 17}{2x^2 - x + 4}$$

CHECK YOUR UNDERSTANDING

Perform long division on the following rational function

$$\frac{2x^3 + x^2 - 4}{x^2 - 1}$$

Complete the following statement

$$\frac{2x^3 + x^2 - 4}{x^2 - 1} = \underline{\hspace{3cm}} + \frac{\underline{\hspace{3cm}}}{x^2 - 1}$$

Method of partial fractions

Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function.

Goal: determine the (indefinite) integral

$$\int f(x)dx$$

CHECK YOUR UNDERSTANDING

Complete the following steps to determine

$$\int \frac{2x^3+x^2-4}{x^2-1}dx$$

You've shown above that

$$\frac{2x^3+x^2-4}{x^2-1} = 2x+1+\frac{2x-3}{x^2-1}$$

Hence, the difficulty lies in determining $\int \frac{2x-3}{x^2-1}dx$.

1. Find constants A and B so that

$$A(x+1)+B(x-1) = 2x-3.$$

2. Observe that $x^2-1=(x-1)(x+1)$. Use the previous problem to complete the following statement

$$\frac{2x^3+x^2-4}{x^2-1} = \underline{\hspace{3cm}} + \frac{\qquad}{x-1} + \frac{\qquad}{x+1}$$

3. Deduce

$$\int \frac{2x^3+x^2-4}{x^2-1}dx =$$

The process of splitting up the rational function $f(x) = \frac{2x^3+x^2-4}{x^2-1}$ into a sum of simpler rational functions is known as the **method of partial fractions**.

Example: Determine

$$\int \frac{x^4 + 2x}{x^2 - 3x + 2} dx$$

First, we perform long division to obtain

Hence,

$$\frac{x^4 + 2x}{x^2 - 3x + 2} = \underline{\hspace{6cm}}$$

and we need to determine

$$\int \underline{\hspace{4cm}} dx$$

Observe that $x^2 - 3x + 2 = \underline{\hspace{2.5cm}}$. We want to find constants A and B such that

$$\frac{\underline{\hspace{2.5cm}}}{x^2 - 3x + 2} = \frac{A}{x-2} + \frac{B}{x-1}$$

Multiplying both sides of this equation by $x^2 - 3x + 2$ gives

$$\underline{\hspace{2.5cm}} = \underline{\hspace{6cm}}$$

We want to determine A, B:

Hence,

$$\frac{\underline{\hspace{2.5cm}}}{x^2 - 3x + 2} = \frac{\underline{\hspace{1.5cm}}}{x-2} - \frac{\underline{\hspace{1.5cm}}}{x-1}$$

Therefore,

$$\int \frac{x^4 + 2x}{x^2 - 3x + 2} dx = \int \underline{\hspace{6cm}} dx$$

$$= \underline{\hspace{7cm}}$$

In tomorrow's lecture we will formalise and generalise the approach we have taken today so that we can handle more complicated rational functions.