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April 16 Lecture

Supplementary References:
- *Single Variable Calculus*, Stewart, 7th Ed.: Section 7.3.
- *Calculus*, Spivak, 3rd Ed.: Section 19

Keywords: inverse trigonometric substitution

Techniques of Integration III. Inverse Trigonometric Substitutions.

Today we investigate the **method of inverse trigonometric substitution**. We will see lots of examples.

Inverse trigonometric substitution

The method of inverse trigonometric substitution proceeds as follows: we are looking to determine

$$\int h(x)dx$$

where $h(x)$ contains one of the following expressions

$$\sqrt{a^2-x^2}, \qquad \sqrt{a^2+x^2}, \qquad \sqrt{x^2-a^2} \qquad \text{where } a \text{ is some constant.}$$

Strategy:

1. Make the following substitution, depending on which of the above expressions appears in $h(x)$:

$x = a\sin(t)$	$\leftrightarrow$	$\sqrt{a^2-x^2}$
$x = a\tan(t)$	$\leftrightarrow$	$\sqrt{a^2+x^2}$
$x = a\sec(t)$	$\leftrightarrow$	$\sqrt{x^2-a^2}$

2. Suppose we've made the substitution $x = T(t)$ above. Write
$$h(T(t))\frac{dx}{dt} = f(t)$$
3. Determine
$$\int f(t)dt$$
4. Substitute $t = T^{-1}(t)$ into the resulting expression.

Useful Trig. Identities

- $\sin^2(t) + \cos^2(t) = 1$
- $\tan^2(t) + 1 = \sec^2(t)$
- $\cos^2(t) = \frac{1}{2}(1 + \cos(2t))$
- $\sin^2(t) = \frac{1}{2}(1 - \cos(2t))$
- $\cos(2t) = \cos^2(t) - \sin^2(t)$
- $\sin(2t) = 2\sin(t)\cos(t)$

CHECK YOUR UNDERSTANDING

Let's determine

$$\int \sqrt{36 - x^2}dx$$

by the method of inverse trigonometric substitution.

1. Let $x = 6\sin(t)$. Show that

$\frac{dx}{dt} = 6\cos(t)$ $\qquad\qquad$ $\sqrt{36 - x^2}\frac{dx}{dt} = 36\cos^2(t)$

$$\sqrt{36-x^2}\cdot\frac{dx}{dt} = \sqrt{6^2(1-\sin^2(t))}\cdot 6\cos(t)$$
$$= 6\cdot\cos(t)\cdot 6\cos(t),$$
$$= 36\cos^2(t)$$

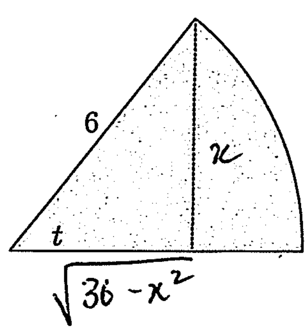
since $1-\sin^2(t) = \cos^2(t)$

2. Determine

$$\int \cos^2(t)dt$$

$$\int \cos^2(t)\,dt = \frac{1}{2}\int (1+\cos(2t))dt$$
$$= \frac{1}{2}\left(t + \frac{\sin(2t)}{2}\right) + C$$
$$= \frac{1}{2}t + \frac{\sin(t)\cos(t)}{2} + C$$

3. Given that $x = 6\sin(t)$, complete the following triangle:



$\cos(t) = \sqrt{36-x^2}$

$\sin(t) = \frac{x}{6}$

4. Combine your answers above to determine

$$\int \sqrt{36 - x^2}dx$$

Hint: you have all the pieces of the puzzle, now see how the **Strategy** *tells you to fit them together.*

$$x = 6\sin(t)$$

$$\int \sqrt{36 - x^2}\, dx = \int 36\cos^2(t)\, dt$$

$$= 18t + 18\cos(t)\sin(t) + C$$

$$= 18\arcsin\left(\frac{x}{6}\right) + 18\,\frac{\sqrt{36-x^2}}{6}\cdot\frac{x}{6} + C$$

$$= 18\arcsin\left(\frac{x}{6}\right) + \frac{\sqrt{36-x^2}\cdot x}{2} + C.$$

Check your understanding

Let's determine

$$\int x\sqrt{x^2 + 1}dx$$

by the method of inverse trigonometric substitution.

1. Let $x = \tan(t)$. Show that

$$\frac{dx}{dt} = \sec^2(t) \qquad x\cdot\sqrt{x^2+1}\frac{dx}{dt} = \tan(t)\sec^3(t)$$

Hint: $\frac{d}{dt}\tan(t) = \sec^2(t) = \frac{1}{\cos^2(t)}$ *and use an appropriate trig. identity*

$$x\sqrt{x^2+1}\,\frac{dx}{dt} = \tan(t)\sqrt{\tan^2(t)+1}\,\sec^2(t)$$

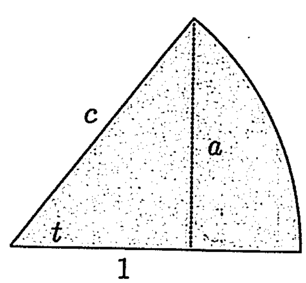
$$= \tan(t)\sqrt{\sec^2(t)}\cdot\sec^2(t) \qquad ; \quad \tan^2(t)+1 = \sec^2(t)$$

$$= \tan(t)\sec^3(t)$$

$$= \frac{\sin(t)}{\cos^4(t)}$$

2. Given that $x = \tan(t)$, use Pythagoras' Theorem to determine a, c.

$$\cos(t) = \frac{1}{\sqrt{1+x^2}}$$



$$a = x$$

$$c = \sqrt{1+x^2}$$

3. Combine your answers above to determine

$$\int x\sqrt{x^2+1}dx$$

$$\int x\sqrt{x^2+1}\,dx = \int \frac{\sin(t)}{\cos^4(t)}\,dt \quad , \quad u = \cos(t), \quad \frac{du}{dt} = -\sin(t)$$

$$= \int -u^{-4}\,du$$

$$= \frac{1}{3}u^{-3} + C \quad = \frac{1}{3}(\cos(t))^{-3} + C$$

$$= \frac{1}{3}\left(1+x^2\right)^{3/2} + C,$$

since $\frac{1}{\sqrt{x^2+1}} = \cos(t)$

Mathematical Workout

Using the substitution $x = 2\sec(t)$ determine

$$\frac{dx}{dt} = 2\sec(t)\tan(t)$$

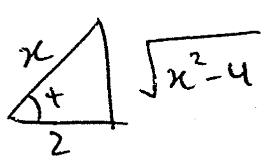
$$\int \frac{\sqrt{x^2-4}}{x}dx$$

$$\frac{\sqrt{x^2-4}}{x}\cdot\frac{dx}{dt} = \frac{\sqrt{4(\sec^2(t)-1)}}{\cancel{2\sec(t)}}\cdot \cancel{2\sec(t)}\tan(t)$$

$$= 2\tan(t)\cdot\tan(t)$$

$$= 2\tan^2(t)$$

$$\Rightarrow \int \frac{\sqrt{x^2-4}}{x}\,dx = \int 2\tan^2(t)\,dt$$



$$= \int 2(1-\sec^2(t))\,dt$$

$$= 2(t - \tan(t)) + C$$

$$= 2\,\mathrm{arcsec}(t) - \sqrt{x^2-4} + C.$$