Middlebury College

Calculus II: Spring 2018

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Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 7.3.
- *Calculus*, Spivak, 3rd Ed.: Section 19

Keywords: inverse trigonometric substitution

Techniques of Integration III. Inverse Trigonometric Substitutions.

Today we investigate the **method of inverse trigonometric substitution**. We will see lots of examples.

Inverse trigonometric substitution

The method of inverse trigonometric substitution proceeds as follows: we are looking to determine

$$\int h(x)dx$$

where $h(x)$ contains one of the following expressions

$$\sqrt{a^2 - x^2}, \qquad \sqrt{a^2 + x^2}, \qquad \sqrt{x^2 - a^2} \qquad \text{where } a \text{ is some constant.}$$

Strategy:

1. Make the following substitution, depending on which of the above expressions appears in $h(x)$:

$x = a\sin(t)$	$\leftrightarrow$	$\sqrt{a^2 - x^2}$
$x = a\tan(t)$	$\leftrightarrow$	$\sqrt{a^2 + x^2}$
$x = a\sec(t)$	$\leftrightarrow$	$\sqrt{x^2 - a^2}$

2. Suppose we've made the substitution $x = T(t)$ above. Write

$$h(T(t))\frac{dx}{dt} = f(t)$$

3. Determine

$$\int f(t)dt$$

4. Substitute $t = T^{-1}(t)$ into the resulting expression.

Useful Trig. Identities

- $\sin^2(t) + \cos^2(t) = 1$
- $\tan^2(t) + 1 = \sec^2(t)$
- $\cos^2(t) = \frac{1}{2}(1 + \cos(2t))$
- $\sin^2(t) = \frac{1}{2}(1 - \cos(2t))$
- $\cos(2t) = \cos^2(t) - \sin^2(t)$
- $\sin(2t) = 2\sin(t)\cos(t)$

Check your understanding

Let's determine

$$\int \sqrt{36 - x^2}dx$$

by the method of inverse trigonometric substitution.

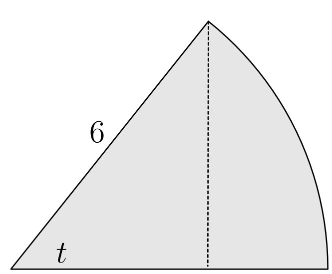
1. Let $x = 6\sin(t)$. Show that

$$\sqrt{36 - x^2}\frac{dx}{dt} = 6\cos^2(t)$$

2. Determine

$$\int \cos^2(t)dt$$

3. Given that $x = 6\sin(t)$, complete the following triangle:



4. Combine your answers above to determine

$$\int \sqrt{36 - x^2} dx$$

*Hint: you have all the pieces of the puzzle, now see how the **Strategy** tells you to fit them together.*

Check your understanding

Let's determine

$$\int \sqrt{x^2 + 1} dx$$

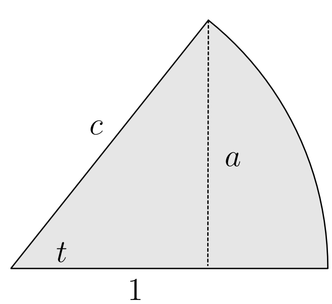
by the method of inverse trigonometric substitution.

1. Let $x = \tan(t)$. Show that

$$\sqrt{x^2 + 1} \frac{dx}{dt} = \cos(t)$$

Hint: $\frac{d}{dt} \tan(t) = \sec^2(t) = \frac{1}{\cos^2(t)}$ *and use an appropriate trig. identity*

2. Given that $x = \tan(t)$, use Pythagoras' Theorem to determine a, c.



3. Combine your answers above to determine

$$\int \sqrt{x^2+1}dx$$

Mathematical Workout

Using the substitution $x = 2\sec(t)$ determine

$$\int \frac{\sqrt{x^2-4}}{x}dx$$