MATH 122 : HW 4/16

Let $f(x) = \cos(x)$.

1a) Since $f^{(n)}(x) = \begin{cases} \pm \cos(x) \\ \pm \sin(x) \end{cases}$, for all n and $-10 \le x \le 10$

we have

$$|f^{(n)}(x)| \le M = 1.$$

Hence, by Taylor's Inequality

$$|R_n(x)| \le \frac{M d^{n+1}}{(n+1)!}, \quad -10 \le x \le 10$$

$$= \frac{10^{n+1}}{(n+1)!} \longrightarrow 0 \quad \text{as } n \to \infty$$

Hence, for any $-10 \le x \le 10$,

since $\lim R_n(x) = 0$, we have

$$f(x) = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

b) Similarly for any $-d \le x \le d$,

$$\lim R_n(x) = 0$$

$\Rightarrow$ $f(x) = \cos(x) =$ Taylor series at x.

c) Hence, for any x, choose $d \ge x$.

Then, $f(x) = $ T.S. by (b).

d) $T_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

$T_6(1) = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} = 0.540277\ldots$

[illegible]

2) a)

$$\frac{\cos(x)-1}{x^2} = \frac{\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\right)-1}{x^2}$$

$$= -\frac{1}{2!}+\frac{x^2}{4!}-\frac{x^4}{6!}+\dots$$

$$\Rightarrow \lim_{x\to 0}\frac{\cos(x)-1}{x^2} = -\frac{1}{2}$$

b)

$$e^{2x} = 1+2x+\frac{(2x)^2}{2!}+(2x)^3$$

$$\frac{x}{e^{2x}-1} = \frac{x}{1+2x+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\dots-1}$$

$$= \frac{\cancel{x}}{\cancel{x}}\cdot\frac{1}{2+\frac{4x}{2!}+\frac{8x^2}{3!}+\dots}$$

$$\Rightarrow \lim_{x\to 0}\frac{x}{e^{x}-1} = \frac{1}{2}.$$

c) Expand $\cos(x)$ as T.S. at c = π:

$$\cos(x) = \cos(\pi) - \frac{\cos(\pi)}{2!}(x-\pi)^2 + \frac{\cos(\pi)}{4!}(x-\pi)^4 + \dots$$

$$= -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \dots$$

$$\Rightarrow \quad \frac{1+\cos(x)}{(x-\pi)^2} = \frac{1+\left(-1+\frac{(x-\pi)^2}{2!}-\frac{(x-\pi)^4}{4!}+\cdots\right)}{(x-\pi)^2}$$

$$= \frac{1}{2!} - \frac{(x-\pi)^2}{4!} + \cdots$$

$$\Rightarrow \lim_{x\to\pi} \frac{1+\cos(x)}{(x-\pi)^2} = \frac{1}{2}.$$

3a) $\int x e^{2x}\, dx$ $\quad f = x \quad g' = e^{2x}$
$\quad f' = 1 \quad g = \frac{1}{2}e^{2x}$

$$= \frac{x}{2}e^{2x} - \frac{1}{2}\int e^{2x}\, dx$$

$$= \frac{xe^{2x}}{2} - \frac{1}{4}e^{2x} + C$$

b) $\int x^2 \sin(3x)\, dx$ $\quad f = x^2 \quad g' = \sin(3x)$
$\quad f' = 2x \quad g = -\frac{\cos(3x)}{3}$

$$= -\frac{x^2}{3}\cos(3x) + \frac{2}{3}\int x\cos(3x)$$

$\quad f = x \quad g' = \cos(3x)$
$\quad f' = 1 \quad g = \frac{\sin(3x)}{3}$

$$= -\frac{x^2}{3}\cos(3x) + \frac{2}{3}\left[\frac{x\sin(3x)}{3} - \frac{1}{3}\int \sin(3x)\, dx\right]$$

$$= -\frac{x^2}{3}\cos(3x) + \frac{2}{9}x\sin(3x) + \frac{2}{27}\cos(3x) + C$$

c) $\int (x^2 + 2x) e^x \, dx$ $\quad f = x^2+2x \quad g' = e^x$

$f' = 2x+2 \quad g = e^x$

$= (x^2+2x)e^x - 2\int (x+1) e^x \, dx$

$= (x^2+2x)e^x - 2\left((x+1) e^x - e^x \right) + C$ $\quad f = x+1 \quad g' = e^x$

$f' = 1 \quad g = e^x$

$= \cancel{x} e^x (x^2) + C = \boxed{x^2 e^x + C}$

d) $\int 1 . (\ln(x))^2 \, dx$ $\quad f = (\ln(x))^2 \quad g' = 1$

$f' = \frac{2 \ln(x)}{x} \quad g = x$

$= x(\ln(x))^2 - 2\int \ln(x) \, dx$ $\quad f = \ln(x) \quad g' = 1$

$f' = \frac{1}{x} \quad g = x$

$= x(\ln(x))^2 - 2\left(x\ln(x) + x \right) + C.$

e) $\int (x+1) \ln(x) \, dx$ $\quad f = \ln(x) \quad g' = x+1$

$f' = \frac{1}{x} \quad g = \frac{x^2}{2} + x$

$= \left(\frac{x^2}{2} + x\right) \ln(x) - \int \left(\frac{x}{2} + 1\right) dx$

$= \left(\frac{x^2}{2} + x\right) \ln(x) - \frac{x^2}{4} - x + C.$

$$\int e^x \cos(x)\,dx \qquad f = e^x \quad g' = \cos(x)$$
$$f' = e^x \quad g = \sin(x)$$

$$= e^x \sin(x) - \int e^x \sin(x)\,dx$$

$$f = e^x \quad g' = \sin(x)$$
$$f' = e^x \quad g = -\cos(x)$$

$$= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx$$

$$\Rightarrow \int e^x \cos(x)\,dx = \frac{1}{2} e^x(\sin(x) + \cos(x)) + C.$$

g) $$\int \arctan(x)\,dx \qquad f = \arctan(x) \quad g' = 1$$
$$f' = \frac{1}{1+x^2} \quad g = x$$

$$= x\arctan(x) - \int \frac{x}{1+x^2}\,dx$$

Let $v = 1+x^2$
$dv = 2x\,dx$

$$= x\arctan(x) - \frac{1}{2}\int \cancel{x}\,\frac{1}{v}\,dv$$
$$= x\arctan(x) - \frac{1}{2}\ln(1+x^2) + C.$$

h) $$\int \frac{\ln(x)}{\sqrt{x}}\,dx \qquad f = \ln(x) \quad g' = \frac{1}{\sqrt{x}}$$
$$f' = \frac{1}{x} \quad g = 2\sqrt{x}$$

$$= 2\sqrt{x}\ln(x) - 2\int \frac{1}{\sqrt{x}}\,dx$$
$$= 2\sqrt{x}\ln(x) - 4\sqrt{x} + C.$$