Middlebury College

Calculus II: Spring 2018

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# April 13 Lecture

Supplementary References:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 4.5, 7.2.
- *Calculus*, Spivak, 3rd Ed.: Section 19.
- *AP Calculus BC*, Khan Academy: Antiderivatives and the fundamental theorem of calculus.

Keywords: $u$-substitution

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## Techniques of Integration II. Substitutions.

Today we review the method of substitution in integration. We apply these methods to trigonometric integrals. In the next lecture we will consider the method of trigonometric *inverse substitution.*

### Integration by substitution

In your previous calculus course you may have seen the technique of integration known as **substitution** or **$u$-substitution**. This technique provides the inverse operation to the **chain rule** for derivatives.

> **Chain Rule**
>
> Let $f(x)$, $g(x)$ be differentiable functions. Then,
>
> $$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$

Check your understanding

Let $f(x) = \sqrt{x}$ and $g(x) = 3x - x^2$.

1. Determine $f(g(x))$.

$$f(g(x)) = \sqrt{3x-x^2}$$

2. Compute $\frac{d}{dx}f(g(x))$.

$$\frac{d}{dx}f(g(x)) = \frac{1}{2} \cdot \frac{3-2x}{\sqrt{3x-x^2}}$$

Using our above computation we can solve the following antiderivative problem: determine

$$\int \frac{1}{2}\frac{3-2x}{\sqrt{3x-x^2}}dx$$

We use the age old process of recognition: we recognise that if $f(x) = \sqrt{x}$ and $g(x) = 3x - x^2$ then

$$\frac{1}{2}\frac{3-2x}{\sqrt{3x-x^2}} = f'(g(x)) \cdot g'(x) = \frac{d}{dx}f(g(x))$$

Thus,

$$\int \frac{1}{2}\frac{3-2x}{\sqrt{3x-x^2}}dx = \int \left(\frac{d}{dx}f(g(x))\right) dx = f(g(x)) = \sqrt{3x-x^2}.$$

We summarise this procedure: suppose we want to solve the antiderivative problem $\int h(x)dx$.

1. Determine $f(x)$ and $g(x)$ such that, if we write $u = g(x)$ then $h(x) = f(u)\frac{du}{dx}$.
2. Determine $\int f(u)du$
3. Substitute $g(x)$ for $u$.

**Example:**

1. Determine

$$\int \cos^3(x)\sin(x)dx$$

We recognise that if we let $u =$ $\cos(x)$ then $u'(x) =$ $-\sin(x)$. Hence,

$$\cos^3(x)\sin(x) = -u^3 \cdot \frac{du}{dx}$$

Here $f(u) =$ $-u^3$ and $\int f(u)du =$ $\frac{-u^4}{4} + C$. Substituting $u =$ $\cos(x)$ we obtain

$$\int \cos^3 \sin(x)dx = \frac{-\cos^4(x)}{4} + C \qquad (*)$$

CHECK YOUR UNDERSTANDING

By differentiating the right hand side of $(*)$, confirm our solution to the antiderivative problem.

2. Determine

$$\int \sin^2(x)\cos^3(x)dx$$

Here we can't use a direct substitution as in the previous problem. However, we will make a clever use of the following trigonometric identity

$$\boxed{\sin^2(x) + \cos^2(x) = 1, \qquad \text{for all } x.}$$

Rearranging we see that we can write $\cos^2(x) = 1 - \sin^2(x)$ and

$$\begin{aligned} \sin^2(x)\cos^3(x) &= \sin^2(x)\cos^2(x)\cos(x) \\ &= \sin^2(x)(1 - \sin^2(x))\cos(x) \\ &= \left(\sin^2(x) - \sin^4(x)\right)\cos(x) \end{aligned}$$

Now, we recognise that if we let $u = \sin(x)$ then $u'(x) = \cos(x)$ and

$$\left(\sin^2(x) - \sin^4(x)\right)\cos(x) = (u^2 - u^4)\frac{du}{dx}$$

Now,

$$\int (u^2 - u^4)du = \frac{u^3}{3} - \frac{u^5}{5} + C.$$

Substituting $u = \sin(x)$ we find

$$\int \sin^2(x)\cos^3(x)dx = \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C$$

CHECK YOUR UNDERSTANDING

1. Determine

$$\int \sin^3(x)dx$$

$$\int \sin^3(x)\,dx = \int (1 - \cos^2(x))\sin(x)dx.$$

$u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$

$\Rightarrow -du = \sin(x)dx$

$$\int -(1-u^2)\,du = -u + \frac{u^3}{3} + C$$

$$= -\cos(x) + \frac{\cos^3(x)}{3} + C$$

2. Determine

$$\int \sin^3(x)\cos^4(x)dx$$

$$\int \sin^3(x)\cos^4(x)\,dx$$

$$= \int \sin(x)(1 - \cos^2(x))\cos^4(x)dx$$

$u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$

$\Rightarrow -du = \sin(x)\,dx$

$$= \int -(u^4 - u^6)\,du$$

$$= \frac{-u^5}{5} + \frac{u^7}{7} + C = -\frac{\cos^5(x)}{5} + \frac{\cos^7(x)}{7} + C$$

Let's determine a general approach to finding

$$\int \sin^n(x)\cos^m(x)dx$$

where $m, n \geq 0$ are integers.

Based on your investigation above, complete the following statements:

- if $n = 2k+1$ is odd, use $\sin^2(x) =$ $1-\cos^2(x)$ to write

$$\sin^{2k+1}(x)\cos^m(x) = (1-\cos^2(x))^k \cos^m(x) \cdot \sin(x)$$

and make the substitution $u =$ $\cos(x)$

- if $m = 2k+1$ is odd, use $\cos^2(x) =$ $1-\sin^2(x)$ to write

$$\sin^n(x)\cos^{2k+1}(x) = \sin^n(x)(1-\sin^2(x))^k \cos(x)$$

and make the substitution $u =$ $\sin(x)$

What if both $m$ and $n$ are even? How can we determine

$$\int \sin^n(x)\cos^m(x)dx$$

in this case?

MATHEMATICAL WORKOUT - FLEX THOSE MUSCLES!

Recall the trigonometric formulae

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$1 = \cos^2(x) + \sin^2(x)$$

Use these formulae to complete the following statements:

- $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$
- $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$

CHECK YOUR UNDERSTANDING

Determine

$$\int \cos^2(x)dx$$

$$\int \cos^2(x)\,dx = \int \frac{1}{2} + \frac{1}{2}\cos(2x)\,dx$$

$$= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$$