Middlebury College

Calculus II: Spring 2018

Contact: `gmelvin@middlebury.edu`

APRIL 11 LECTURE SUPPLEMENTARY REFERENCES:

- *Single Variable Calculus*, Stewart, 7th Ed.: Section 4.5, 7.1.
- *Calculus*, Spivak, 3rd Ed.: Section 19.
- *AP Calculus BC*, Khan Academy: Antiderivatives and the fundamental theorem of calculus.

KEYWORDS: antiderivative, elementary function, integration by parts

---

THE ANTIDERIVATIVE PROBLEM. TECHNIQUES OF INTEGRATION I.

Today we review the antiderivative problem and begin extending our techniques of (indefinite) integration.

The antiderivative problem

For the next few lectures we are going to investigate the following

Antiderivative problem:

Let $f(x)$ be a function. Does there exist a differentiable function $F(x)$ satisfying

$$\frac{d}{dx}F(x) = f(x).$$

CHECK YOUR UNDERSTANDING

Let $f(x) = \ln(x)$, where $\ln(x)$ is the inatural logarithm function. Recall that $\frac{d}{dx}\ln(x) = \frac{1}{x}$. Show that $F(x) = x\ln(x) - x$ provides a solution to the antiderivative problem for $f(x)$.

If $f(x)$ is a continuous function then the Fundamental Theorem of Calculus provides the general solution to the antiderivative problem:

Fundamental Theorem of Calculus

Let $f(x)$ be a continuous function defined on the closed interval $a \leq x \leq b$. Then, the function

$$F(x) = \int_a^x f(u)du$$

is an antiderivative of $f(x)$.

The Fundamental Theorem of Calculus tells us that **any** continuous function admits an antiderivative $F(x)$. However, while we are given a recipe for determining an antiderivative $F(x)$ - i.e. 'simply' determine the area below the graph of $f(u)$ as a function of the upper limit of integration $u = x$ - it's not so easy to *know* whether we can recognise $F(x)$.

Question:

Let $f(x)$ be a continuous function. Is there a systematic way to describe the antiderivative $F(x)$ in terms of well-known functions (e.g. rational functions, trigonometric functions, exp, log etc)?

We make precise our notion of a '*well-known*' function: a function $f(x)$ is an **elementary function** if it can be obtained by addition, multiplication, division, and composition from the rational functions (i.e. fractions involving polynomial functions), the trigonometric functions and their inverses, and the inverse functions ln, exp.

Remark: for the most part, essentially all functions you have encountered in your mathematical life are elementary functions.

We will spend the next few Lectures introducing a variety of techniques that will help us find solutions to the following problem.

Question:

Let $f(x)$ be an elementary function. Is it possible to describe the antiderivative $F(x)$ as an elementary function?

Remark:

1. In general, it is not possible to describe the antiderivative of an elementary function as an elementary function. For example, a difficult Theorem states the following:

   There is no elementary function $F(x)$ such that $\dfrac{d}{dx}F(x) = \exp(-x^2)$.

   The proof of this result relies on advanced mathematics far beyond the scope of Calculus II. However, remember that the Fundamental Theorem of Calculus shows that $\exp(-x^2)$ does admit an antiderivative; the above statement says that this antiderivative is not an elementary function.

2. The **indefinite integral**

$$\int f(x)dx$$

is an expression used in place of '*the (general) antiderivative* $F(x)$ *of* $f(x)$'. In particular, the problem

$$\textit{Determine} \int f(x)dx$$

is the same problem as

Find/determine the (general) antiderivative $F(x)$ of $f(x)$.

The Fundamental Theorem of Calculus tells us that we solve this problem using *integration*. As such, an '*indefinite integration problem*' is synonymous with an '*antiderivative problem*'.

Integration by parts

Recall the Product Rule for derivatives: let $f(x)$, $g(x)$, be differentiable functions.

Product Rule

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

Taking antiderivatives of both sides of the above formula gives

Integration by parts

$$\int fg' = \underline{\hspace{3cm}} - \int \underline{\hspace{3cm}} \qquad (*)$$

Integration by parts provides us with the following strategy to determine the integral $\int h(x)dx$:

Strategy: Find specific functions $f(x)$ and $g(x)$ satisfying

(A) $f(x)g'(x) = h(x)$, and so that

(B) it's easier to determine $\int f'(x)g(x)dx$.

(C) Now, use the integration by parts formula $(*)$.

Example:

1. Determine

$$\int x\sin(x)dx$$

We proceed by integration by parts.

Try: $f(x) = x$, and $g'(x) = \sin(x)$. Then, $f'(x) = 1$ and $g(x) = -\cos(x)$. Hence, using integration by parts $(*)$

$$\begin{aligned}\int x\sin(x)dx &= x(-\cos(x)) - \int 1.(-\cos(x))dx \\ &= -x\cos(x) + \int \cos(x)dx \\ &= -x\cos(x) + \sin(x) + C\end{aligned}$$

Check your understanding

Verify that

$$\frac{d}{dx}(-x\cos(x) + \sin(x)) = x\sin(x).$$

2. Sometimes we may have to integrate by parts several times. For example, consider the following antiderivative problem: Determine

$$\int x^2 \exp(x)dx$$

Try: $f(x) = x^2$, $g'(x) = \exp(x)$. Then, $f'(x) = 2x$ and $g(x) = \exp(x)$. Hence,

$$\int x^2 \exp(x)dx = \rule{6cm}{0.4pt}$$

To determine ____________________ we integrate by parts again.

Try: $f(x) =$ __________, $g'(x) =$ __________. Then, $f'(x) =$ __________ and $g(x) =$ __________. Hence,

$$\rule{4cm}{0.4pt} = \rule{4cm}{0.4pt}$$

$$= \rule{6cm}{0.4pt}$$

Hence,

$$\int x^2 \exp(x)dx = \rule{8cm}{0.4pt}$$

(*Recall that the sign of the constant of integration C is irrelevant*)

3. Sometimes we may have to be a bit clever with our choice of $f(x)$ and $g'(x)$. Consider the antiderivative problem: determine

$$\int \log(x)dx$$

Try: $f(x) = \log(x)$, $g'(x) = 1$. Then, $f'(x) = \frac{1}{x}$ and $g(x) = x$. Hence, using integration by parts

$$\begin{aligned}\int \log(x)dx &= \log(x)x - \int \frac{1}{x}.xdx \\ &= \log(x)x - \int 1.dx \\ &= \log(x)x - x + C\end{aligned}$$

4. Sometimes we may want to use integration by parts to find $\int h$ in terms of $\int h$ again, and then solve for $\int h$. Consider the following antiderivative problem: Determine

$$\int \frac{\log(x)}{x} dx$$

Try: $f(x) = \log(x)$, $g'(x) = \frac{1}{x}$. Then, $f'(x) = \frac{1}{x}$ and $g(x) = \log(x)$. Hence,

$$\begin{aligned}\int \frac{\log(x)}{x} dx &= \log(x)\log(x) - \int \frac{1}{x}.\log(x)dx \\ &= (\log(x))^2 - \int \frac{\log(x)}{x} dx \\ \implies 2\int \frac{\log(x)}{x} &= (\log(x))^2\end{aligned}$$

Hence,

$$\int \frac{\log(x)}{x} dx = \frac{1}{2}(\log(x))^2 + C$$