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Wallis's Product, Brouncker's Continued Fraction, and Leibniz's Series

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Abstract

A historical sketch is given of WALLIS's infinite product for $4/\pi$, and of the attempts which have been made, over more than three centuries, to find the method by which BROUNCKER obtained his equivalent continued fraction. A derivation of BROUNCKER'S formula is given. Early results obtained by Indian mathematicians for the series for $\pi/4$, later named for LEIBNIZ, are reviewed and extended. A conjecture is made concerning BROUNCKER'S method of obtaining close bounds for π.

1. Wallis's Product

In 1656, the largely self-taught Oxford mathematician, JOHN WALLIS (1616–1703) published his greatest work, the *Arithmetica Infinitorum*. By reformulating and systematizing the largely geometric methods of his predecessors, particularly J. KEPLER, R. DESCARTES and B. CAVALIERI, in (what would now be called) more analytic terms, he was able to develop techniques which permitted the quadrature and cubature of certain classes of curves and surfaces. His general mode of procedure, perhaps stemming from his experience in cryptanalysis (as a practitioner of which he was one of the most outstanding in history), was to rely upon analogy and induction. From particular numerical examples, supplemented perhaps by analogical extensions, heuristic rules would be developed which would later be formalized as propositions, without deductive proofs. His bold inductive approach, coupled with his generally correct mathematical intuition, led to numerous interesting results, and had considerable influence on his successors, including ISAAC NEWTON and LEONHARD EULER.

The last part of his book is devoted to the millenia-old problem of the quadrature of the circle,[1] and culminates in an expression for the reciprocal of the ratio

[1] Some extracts, in English, are given in *A Source Book in Mathematics, 1200–1800*, edited by D. J. STRUIK, Cambridge, Mass., 1969.

of the area of a circle to that of the square of a diameter which may be put as

$$\frac{4}{\pi}=\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{7}{8}\cdots\Big] \tag{1.1}$$

in which the partial products are alternately in excess and defect.[2]

Within a few decades, WALLIS's methods were largely superseded by the more efficient methods of the calculus, independently developed by NEWTON and G. W. LEIBNIZ. Nowadays, WALLIS is seldom mentioned in standard texts on advanced calculus, except in connection with the determination of the constant in JAMES STIRLING's approximation for factorial n. Relatively little known, however, is the fact that WALLIS's formula (1.1) can be used to derive Stirling's approximation.[3]

2. Brouncker's Continued Fraction

Before the completion of his book, WALLIS induced his friend, Lord WILLIAM BROUNCKER (1620?–1684), afterwards the first president of the Royal Society, to investigate his expression (1.1) for $4/\pi$. BROUNCKER'S reply was in the form of (the reciprocal of) a continued fraction expansion

$$\frac{\pi}{4}=\frac{1}{1+}\,\frac{1}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+}\cdots\frac{(2n-1)^2}{+2+}\cdots \tag{2.1}$$

which he gave without proof (and some numerical approximations which will be discussed in Section 4). WALLIS attempted to outline a method by which he thought BROUNCKER'S result had been obtained.[4]

It is convenient at this point, before continuing the historical narrative further, to discuss a closely related infinite series representation for $\pi/4$, and to develop some properties of the continued fraction in (2.1) which will subsequently prove useful.

In 1671, JAMES GREGORY in a letter to JOHN COLLINS[5] gave the equivalent of the series

$$\operatorname{arc\,tan} x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots \tag{2.2}$$

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[2] As early as 1593, an infinite product expansion for $\pi/2$ in terms of a sequence of square roots had been given by FRANÇOIS VIÈTE. But WALLIS's remarkable expression of a purely geometric ratio in terms of natural numbers in a Greek fret pattern, aroused considerable interest. About 1730, it became for EULER the starting point for his development of the gamma function.

[3] J.-A. SERRET, "Sur l'évaluation approchée du produit $1\cdot2\cdot3\cdot\ldots\cdot x$, lorsque x est un très-grands nombre, et sur la formule de Stirling," *Comptes Rendus de l'Académie des Sciences*, T. 50, (1860), 662–666. H. JEFFREYS & B. SWIRLES, *Methods of Mathematical Physics*, 3rd ed., Cambridge, 1956, 468.

[4] See Proposition CXCI of the *Arithmetica Infinitorum* and what follows.

[5] See *James Gregory Tercentenary Memorial Volume*, edited by H. W. TURNBULL, London, 1939, 168–176.

($-1 \leqq x \leqq 1$), which he presumably obtained by the equivalent of integrating $1/(1+x^2)$. (This was just one of many notable results obtained by him, and gradually disseminated among a small group of British mathematicians, but not published until long after his premature death in 1675.) Sometime during the autumn of 1673, LEIBNIZ, in the course of some researches on the quadrature of segments of conic sections[6], obtained the series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \tag{2.3}$$

and communicated it to CHRISTIAAN HUYGENS and other friends. It was not until 1675 that he learned from HENRY OLDENBURG, one of the first secretaries of the Royal Society, that his series was only a particular case of that obtained previously by GREGORY.[7]

If a method of EULER for transforming a series into an equivalent continued fraction[8] is applied to the series in (2.3), a continued fraction is obtained which is identical with that in (2.1), and thus a proof of BROUNCKER's result follows.

But for later application in Section 4, it is convenient to obtain directly the relations between the partial convergents of the continued fraction and the partial sums of the series.

The numerator and denominator of the n^{th} partial convergent of the continued fraction in (2.1) satisfy the recursion equations

$$\left.\begin{aligned} p_n &= 2p_{n-1} + (2n-3)^2\, p_{n-2} \\ q_n &= 2q_{n-1} + (2n-3)^2\, q_{n-2} \end{aligned}\right\}, \quad n \geqq 2, \tag{2.4}$$

respectively, with initial conditions

$$p_0 = 0, \quad p_1 = 1; \quad q_0 = 1, \quad q_1 = 1. \tag{2.5}$$

From this it can be verified that

$$\left.\begin{aligned} p_n &= (2n-1)\, p_{n-1} + (-1)^{n-1}\,(2n-3)!! \\ q_n &= (2n-1)!! \end{aligned}\right\}, \quad n \geqq 2,$$

and that

$$\left.\begin{aligned} p_n &= (2n-1)!!\left(1 - \frac{1}{3} + \frac{1}{5} - \dots + \frac{(-1)^{n-1}}{2n-1}\right) \\ q_n &= (2n-1)!! \end{aligned}\right\}, \quad n \geqq 1.$$

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[6] An absorbing account describing the mathematical researches of LEIBNIZ (and others), during a very creative period, is given in a book by J. E. HOFMANN, *Leibniz in Paris 1672–1676*, Cambridge, 1974.

[7] A history of earlier developments of this series by Indian mathematicians is outlined in Section 4.

[8] See *e.g.*, A. N. KHOVANSKI, *The Application of Continued Fractions ...*, Groningen, 1963, 14–15.

Thus for $n \geqq 1$, the relations sought are

$$\frac{p_{n+1}}{q_{n+1}} = \frac{1}{1+}\frac{1^2}{2+}\frac{3^2}{2+}\cdots+\frac{(2n-1)^2}{2} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots + \frac{(-1)^n}{2n+1} = S_n. \quad (2.6)$$

Let $n \to \infty$ and apply (2.3) to get (2.1).

How did BROUNCKER actually obtain (2.1)? This question has intrigued mathematicians and historians for more than three centuries. EULER's proof and that developed above depend on a knowledge of LEIBNIZ's series—a result which was obtained years after BROUNCKER had stated (2.1), and which presumably was unknown to him when he investigated WALLIS's product (1.1).

As mentioned above, after receiving BROUNCKER's result WALLIS attempted to develop a proof in which the key idea is the following theorem:

If a is a positive integer, $a \geqq 2$, and $K(x)$ is the continued fraction,

$$K(x) = x + \frac{1}{2x+}\frac{3^2}{2x+}\frac{5^2}{2x+}\cdots, \quad \textit{then} \quad a^2 = K(a-1)\cdot K(a+1). \quad (2.7)$$

WALLIS seems to have obtained this by induction from some numerical results. If (1.1) is rewritten in the form

$$\frac{4}{\pi} = \lim_{n\to\infty} \frac{3\cdot 3\cdot 5\cdot 5\ldots(2n+1)\cdot(2n+1)}{2\cdot 4\cdot 6\cdot 6\ldots(2n)\cdot(2n+2)}$$

$$= \lim_{n\to\infty} 2^2\,\frac{6\cdot 6\cdot 10\cdot 10\ldots(4n+2)\cdot(4n+2)}{4\cdot 4\cdot 8\cdot 8\ldots(4n)\cdot(4n)}\,\frac{1}{4n+4},$$

and a in (2.7) is replaced by $2, 4, 6, \ldots, 4n+2$, successively, since $\lim_{n\to\infty} K(4n+3)/(4n+4) = 1$, one gets

$$\frac{4}{\pi} = \lim_{n\to\infty} K(1)\cdot K(3)\,\frac{K(5)\cdot K(7)}{K(3)\cdot K(5)}\,\frac{K(9)\cdot K(11)}{K(7)\cdot K(9)}$$

$$\cdots\frac{K(4n+1)\cdot K(4n+3)}{K(4n-1)\cdot K(4n+1)}\,\frac{1}{4n+4} = K(1),$$

i.e., BROUNCKER's result.

Later, EULER, though he believed it improbable that this was the method by which BROUNCKER obtained (2.1),[9] repeatedly but unsuccessfully tried to prove (2.7).[10] It was not until more than two centuries after WALLIS stated his theorem that G. BAUER,[11] using methods of the theory of determinants, proved a general theorem which includes (2.7) as a particular case.

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[9] See L. EULER, "De transformatione seriei divergentis ...," *Opera Omnia*, Ser. I, T. 16, P. 1, Zürich, 1933, 34–46, particularly 43–44.

[10] A summary of EULER's related work is given by A. SPEISER in G. FABER's "Uebersicht über die Bände ...," T. VI, XCVII–CV, in *Opera Omnia*, Ser. I, T. **16**, P. 2, Basel, 1935.

[11] Von einem Kettenbruche Euler's und einem Theorem von Wallis." *Abh. d. II Cl. d. Bayerischen Akad. der Wiss.* Bd. **11**, Abt. 2 (1872), 96–116.

3. A Derivation of Brouncker's Continued Fraction

In the late nineteenth and early twentieth centuries, in their works on the history of mathematics, R. REIFF and M. CANTOR evaluated WALLIS's method of proof and concluded that it was too artificial to be the manner in which BROUNCKER obtained his result—a view which was also espoused by A. VON BRAUNMÜHL and J. TROPFKE. But about the middle of this century, contrary opinions were expressed by V. BRUN,[12] J. E. HOFMANN[13], and D. T. WHITESIDE[14] who gave proofs and discussions of results associated with (2.7). Pending the discovery of new evidence, no resolution of this question can be made.[15]

A derivation of BROUNCKER'S continued fraction is given here. It consists of (i) transforming an infinite product expansion (3.3), which includes an equivalent of WALLIS'S, into an infinite series of partial fractions (3.5), (ii) developing a recursion formula for an auxiliary function (3.8) associated with the sum of partial fractions (3.7), (iii) using the recursion formula to obtain a continued fraction (3.11), from which BROUNCKER'S result follows. While the details of this derivation are presented in modern form—beyond the ken of seventeenth century mathematicians—the writer believes that an analyst of BROUNCKER'S caliber could have developed the formal steps outlined above.

WALLIS'S product (1.1) is sometimes written as

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2\cdot 2}{1\cdot 3}\,\frac{4\cdot 4}{3\cdot 5}\,\frac{6\cdot 6}{5\cdot 7}\cdots\frac{(2n)\cdot(2n)}{(2n-1)\cdot(2n+1)}. \tag{3.1}$$

An expression which includes this will be obtained as the limit of a finite product. Since for $|x| < 1$, $\Gamma(1+n+x) = \Gamma(1+x)\prod_{k=1}^{n}(k+x)$,

$$f_n(t) = \frac{[(2n)!!]^2}{(2^2-t^2)(4^2-t^2)\dots[(2n)^2-t^2]} = \frac{\Gamma(1-t/2)\cdot\Gamma(1+t/2)\,(n!)^2}{\Gamma(1+n-t/2)\cdot\Gamma(1+n+t/2)}. \tag{3.2}$$

Now $\Gamma(1-t/2)\cdot\Gamma(1+t/2) = (\pi t/2)/\sin(\pi t/2)$. Thus

$$\begin{aligned} f(t) = \lim_{n\to\infty} f_n(t) &= \frac{(\pi t/2)}{\sin(\pi t/2)} \lim_{n\to\infty} \frac{(n!)^2}{\Gamma(1+n-t/2)\cdot\Gamma(1+n+t/2)} \\ &= 1\Big/\prod_{k=1}^{\infty}\left(1-\frac{t^2}{4k^2}\right), \end{aligned} \tag{3.3}$$

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[12] V. BRUN, "Wallis's og Brouncker's formler for π", *Nordisk Matematisk Tidsskrift* Bind **33** (1951), 73–81.

[13] J. E. HOFMANN, "Über Brounckers Kettenbruchentwicklung für Quadratzahlen," *Monatsber. der Akad. der Wiss. Berlin*, Bd. **2** (1960), 310–314.

[14] D. T. WHITESIDE, "Patterns of Mathematical Thought in the later Seventeenth Century," *Archive for History of Exact Sciences*, Vol. **1** (1960), 179–388. See particularly 210–213.

[15] In a footnote on p. 212 of the preceding reference, some unpublished letters of WALLIS to BROUNCKER are mentioned which may contain new information.

which for $t = 1$ yields (3.1). The partial fraction decomposition of $f_n(t)$ is

$$f_n(t) = \sum_{r=1}^{n} \frac{1}{r^2 - \frac{t^2}{4}} \frac{(n!)^2}{\prod_{s \neq r} (s^2 - r^2)},$$

where the product in the denominator on the right is composed of the factors

$$\prod_{s=1}^{r-1} (s - r)(s + r) = (-1)^{r-1} (r - 1)!\,(2r - 1)!/r!,$$

$$\prod_{s=r+1}^{n} (s - r)(s + r) = (n - r)!\,(n + r)!/(2r)!.$$

Thus

$$f_n(t) = 2 \sum_{r=1}^{n} \frac{(-1)^{r-1} n!\, n!}{(n - r)!\,(n + r)!} \frac{(2r)^2}{(2r)^2 - t^2}$$

$$= 1 + \sum_{r=1}^{n} \frac{(-1)^{r-1} n!\, n!}{(n - r)!\,(n - r)!} \frac{2t^2}{(2r)^2 - t^2}. \tag{3.4}$$

Let $n \to \infty$. The series on the right in (3.4) becomes the DE LA VALLÉE POUSSIN sum of a convergent series.[16] Thus $f(t)$ can be represented as the sum of a series of partial fractions[17]

$$f(t) = 1 + \sum_{r=1}^{\infty} \frac{(-1)^r 2t^2}{(2r)^2 - t^2} = 1 + \frac{t}{2} \sum_{s=1}^{\infty} \frac{(-1)^s}{s + \frac{t}{2}} + \frac{t}{2} \sum_{s=0}^{\infty} \frac{(-1)^s}{s + t - \frac{t}{2}} \tag{3.5}$$

or, in terms of the function,[18]

$$\beta(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{x + r}; \quad x \neq 0, -1, -2, \ldots, \tag{3.6}$$

$f(t)$ takes the form

$$f(t) = \frac{t}{2} \cdot \left[\beta\left(\frac{t}{2}\right) + \beta\left(1 - \frac{t}{2}\right)\right]. \tag{3.7}$$

A recursion formula for $x \cdot \beta(x)$, from which a continued fraction can be developed, can readily be obtained. From (3.10),

$$x \cdot \beta(x) + x \cdot \beta(x + 1) = 1, \quad (x + 1) \cdot \beta(x + 1) + (x + 1) \cdot \beta(x + 2) = 1,$$

whence

$$x \cdot \beta(x) = \beta(x + 1) + (x + 1) \cdot \beta(x + 2) \tag{3.8}$$

[16] See *e.g.*, G. H. HARDY, *Divergent Series*, Oxford, 1949, 88, 92–93.

[17] A direct but non-rigorous transformation of an infinite product, similar to $f(t)$, into the sum of an infinite series of partial fractions was given by K. H. SCHELLBACH, "Die einfachsten periodischen Functionen," *Jour. f. d. r. u. ang. Math.*, Bd. **48** (1854), 207–236.

[18] See N. NIELSEN, *Handbuch der Theorie der Gammafunktion*, repr. Chelsea, 1965, 16.

and

$$x\frac{\beta(x)}{\beta(x+1)} = 1 + \cfrac{(x+1)^2}{(x+1)\dfrac{\beta(x+1)}{\beta(x+2)}}.$$

On substituting for x the values $x+1, x+2, \ldots, x+n-1$, successively in the last equation, one gets

$$x\frac{\beta(x)}{\beta(x+1)} = 1 + \frac{(x+1)^2}{1+}\,\frac{(x+2)^2}{1+}\,\frac{(x+3)^2}{1+}\cdots\frac{(x+n)^2}{+(x+n)\dfrac{\beta(x+n)}{\beta(x+n+1)}}.$$

Now $x\cdot\beta(x)/\beta(x+1) = -x + 1/\beta(x+1)$. On substituting this in the last equation and then replacing x by $x-1$ and rewriting, one gets

$$x\cdot\beta(x) = \frac{1}{1+}\,\frac{x}{1+}\,\frac{(x+1)^2}{1+}\cdots\frac{(x+n-1)^2}{+(x+n-1)\dfrac{\beta(x+n-1)}{\beta(x+n)}}. \tag{3.9}$$

By a theorem of PRINGSHEIM[19], the continued fraction

$$\frac{1}{1+}\,\frac{x}{1+}\,\frac{(x+1)^2}{1+}\cdots\frac{(x+n)^2}{1+}\cdots$$

converges for $x > 0$. Thus for $x > 0$,

$$x\cdot\beta(x) = \frac{1}{1+}\,\frac{x}{1+}\,\frac{(x+1)^2}{1+}\,\frac{(x+2)^2}{1+}\,\frac{(x+3)^2}{1+}\cdots. \tag{3.10}$$

Substitute $x = \frac{1}{2}$ in the last equation and simplify. Then

$$\frac{1}{2}\beta\left(\frac{1}{2}\right) = \frac{1}{1+}\,\frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+}\,\frac{7^2}{2+}\cdots \tag{3.11}$$

and from (3.7), and (3.1), one has the equivalent of BROUNCKER'S continued fraction.

4. Leibniz's Series

The series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots + \frac{(-1)^n}{2n+1} + \ldots \tag{4.1}$$

which has already been mentioned in Section 2, is, to paraphrase LEIBNIZ, perhaps the simplest theoretical formula involving π which has been obtained. But the direct use of this series for the computation of approximations to π is impractical, for the error involved in approximating $\pi/4$ by the n^{th} partial sum of the series

[19] See (8), p. 45.

is numerically less than $1/(2n+1)$. Hence, to obtain m correct decimal places in an approximation, a partial sum of at least $5 \cdot 10^{m-1}$ terms must be evaluated. Because of the slow convergence of (4.1), and of the alternating harmonic series for $\ln 2$, for more than three centuries these have become favorite test examples for writers who wished to exhibit the efficacy of techniques developed by them for accelerating the convergence of series and sequences.

In 1835 C. M. WHISH, of the East India Company, described the quadrature of the circle and related infinite series which appeared in four Hindu Sastras.[20] He quoted the remarkable proportion, equivalent to 104348 : 33215, as an approximation to π. (This ratio, which is correct to nine decimals and errs in excess, is equal to the sixth partial convergent in the continued fraction expansion of π.) He also mentioned various series for π, including the equivalent of (4.1) and associated convergence factors. The earliest of the series developed by the Indian mathematicians appears to date from the beginning of the sixteenth century.

Although WHISH's memoir was noted by S. RIGAUD,[21] the work of the Indian mathematicians was not noticed in any of the standard histories of mathematics for more than a century. In 1944, K. M. MARAR & C. T. RAJAGOPAL wrote "a sequel" to WHISH's memoir[22] which was followed up by three expository articles published in *Scripta Mathematica* in 1949, 1951, and 1952. The work of the Indian mathematicians was reviewed by JOS. E. HOFMANN,[23] who also gave plausible reconstructions of empirical bases for some of their results.

As approximations to $\pi/4$, the Indian mathematicians gave the equivalents of

$$N_n = \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} + T_n; \quad n = 1, 2, 3, \ldots \tag{4.2}$$

where T_n is a correction term. For T_n, the increasingly accurate functions

$$\frac{(-1)^n}{4n}, \frac{(-1)^n n}{4n^2+1}, \frac{(-1)^n n^2+1}{n(4n^2+5)}, \tag{4.3}$$

were given. From this follows the series

$$N_1 + \sum_{k=1}^{\infty} (N_{k+1} - N_k), \tag{4.4}$$

which converges much more rapidly to $\pi/4$ than (4.1).

(The idea of approximating the sum of an infinite series by an n^{th} partial sum plus a correction term was used by ARCHIMEDES in his treatise on the quadrature

[20] "On the Hindú Quadrature of the Circle, and the infinite Series of the proportion of the circumference to the diameter exhibited in the four Sástras, the Tantra Sangraham, Yucti Bhásá, Carana Padhati, and Sadratnamála," *Trans. of the Royal Asiatic Society of Great Britain and Ireland*, Vol III (1835), 509–523.

[21] *Correspondence of Scientific Men of the Seventeenth Century*, Oxford, 1841, Vol. 2, 590–591.

[22] "On the Hindu Quadrature of the Circle," *Jour. of the Bombay Branch of the Royal Asiatic Society*. N.S. Vol. **20** (1944), 65–82.

[23] "Über eine altindische Berechnung von π und ihre allgemeine Bedeutung," *Mathematisch-Physikalische Semesterberichte*. Bd. **3** (1953), 193–206.

of the parabola.[24] He obtained the equivalent of a sequence $\{A_k\}$ with $A_k = 4A_{k+1}$; $k = 1, 2, 3, \ldots$, and then showed that $\sum_{k=1}^{\infty} A_k$ could not be greater or less than $\sum_{k=1}^{n} A_k + A_n/3 = 4A_1/3$.

Two millenia later, NEWTON, in his letter of 24 October 1676 to OLDENBURG for LEIBNIZ, considered the problem of accelerating the convergence of (4.1).[25] He proposed the equivalent of taking $T_n = (-1)^n/2/(2n+1)$ in (4.2), and he also suggested the use of the simpler series (from a computational standpoint),

$$1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \frac{1}{25} - \frac{1}{31} + \frac{1}{33}, \text{ etc.,}$$

which is to (4.1) as $1 + \sqrt{2}$ is to 2. This series is the arithmetic mean of (4.1) and

$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \ldots = \int_0^1 \frac{1+x^2}{1+x^4}\,dx = \frac{1}{2}\int_{-1}^{1} \frac{dx}{1 + x\sqrt{2} + x^2}.$$

There is no evidence that the work of the Indian mathematicians outlined above was known in Europe in the seventeenth century, but, in his investigation initiated at WALLIS's request, BROUNCKER encountered a closely related problem. The continued fraction (2.1) given in the *Arithmetica Infinitorum* is as it stands, of little value for computational purposes. But directly following the statement of this, some continued fraction approximations are given whose significance appears to have been overlooked in the standard works on the history of mathematics in this period. These are

$$1,\ 1 + \frac{1}{3},\ 1 + \frac{1}{2+}\,\frac{9}{5},\ 1 + \frac{1}{2+}\,\frac{9}{2+}\,\frac{25}{7}, \quad \text{and} \quad 1 + \frac{1}{2+}\,\frac{9}{2+}\,\frac{25}{2+}\,\frac{49}{9}, \tag{4.5}$$

and are alternately smaller and greater than WALLIS's ratio $4/\pi$. On rewriting this, one finds a general approximation to $\pi/4$, essentially equivalent to (4.2), in the form of a continued fraction

$$\frac{1}{1+}\,\frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+}\cdots + \frac{(2n-1)^2}{2 + u(n)}, \tag{4.6}$$

where, in this case, $u(n) = 2n - 1$. Thus on substituting, one gets from (2.6)

$$\frac{p_{n+1} + (2n-1)\,p_n}{q_{n+1} + (2n-1)\,q_n} = \frac{S_n + \dfrac{2n-1}{2n+1} S_{n-1}}{1 + \dfrac{2n-1}{2n+1}} = S_{n-1} + \frac{(-1)^n}{4n}, \tag{4.7}$$

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[24] *The Works of Archimedes*, edited by T. L. HEATH, repr. Dover, 1953, cxliii, and 233–252.

[25] *The Correspondence of Isaac Newton*, Volume II, 1676–1687, edited by H. W. TURNBULL, Cambridge, 1960, pp. 140, 156.

which is identical to the first of the approximations to $\pi/4$ found by the Indian mathematicians in (4.3). BROUNCKER has, in effect, in (4.5), introduced a convergence factor in a continued fraction—a concept which was apparently not used again until rediscovered by J. J. SYLVESTER in connection with a closely related problem, more than two centuries later.[26]

A general result for the remainder after n terms in (4.1) and for $u(n)$ in (4.6) is readily obtained:

$$\frac{\pi}{4} = \frac{1}{2}\beta\left(\frac{1}{2}\right) = \frac{1}{2}\left[\beta\left(\frac{1}{2}\right) - (-1)^n\beta\left(n+\frac{1}{2}\right)\right] + \frac{(-1)^n}{2}\beta\left(n+\frac{1}{2}\right). \quad (4.8)$$

The continued fraction expansion of the remainder is[27]

$$\frac{(-1)^n}{2}\beta\left(n+\frac{1}{2}\right) = \frac{(-1)^n}{2}\left[\frac{1}{2n+}\,\frac{1^2}{2n+}\,\frac{2^2}{2n+}\,\frac{3^2}{2n+}\cdots\right], \quad (4.9)$$

in which the first three convergents are given in (4.3), and it also follows in (4.6) that

$$u(n) = \frac{2}{\beta\left(n+\frac{1}{2}\right)} - (2n+1) = (2n-1) + 2\left[\frac{1^2}{2n+}\,\frac{2^2}{2n+}\,\frac{3^2}{2n+}\cdots\right]. \quad (4.10)$$

(The above procedure can, in principle, yield arbitrarily accurate approximations, but its computational utility is limited. For this purpose, *e.g.* the substitutions of the series[28]

$$\frac{(-1)^n}{2}\beta\left(n+\frac{1}{2}\right) = \frac{(-1)^n}{2}\sum_{k=0}^{\infty}\frac{k!}{(2n+1)(2n+3)\ldots(2n+2k+1)}$$

is preferable.)

A remarkable geometric representation of LEIBNIZ's series (4.1) was given by V. BRUN.[29] The relation of the series to a problem in number theory is discussed by W. SIERPINSKI.[30]

5. Brouncker's Computation of π

After the publication of WALLIS's *Arithmetica Infinitorum*, various (valid) criticisms were made by CHRISTIAAN HUYGENS and PIERRE DE FERMAT. WALLIS replied to HUYGENS,[31] and similarly to Sir KENELM DIGBY[32] (the intermediary

[26] "Note on a new continued fraction applicable to the quadrature of the circle." *Collected Mathematical Papers* ..., Vol. II (1854–1873), 691–693.

[27] See, *e.g.* N. E. NÖRLUND, "Fractions continues et différences réciproques," *Acta Mathematica*, T. 34 (1911), 106 (34).

[28] See footnote 18, p. 246 (7).

[29] V. BRUN, "Leibniz' formula for π deduced by a "mapping" of the circular disc," *Nordisk Matematisk Tidskrift*, Bind 18 (1970), 73–81.

[30] *Elementary Theory of Numbers*, Warsaw, 1964, 356–357 and 434.

[31] *Oeuvres Complètes de Christiaan Huygens*, T. 1, La Haye (1888), 476–480 and 494–495. (Letters of 22 August and September 1656.)

[32] *Commercium Epistolicum*, Oxford, 1658. (Letter V of 6 June 1657.)

in his correspondence with FERMAT). To the latter he wrote, "I am not too disquieted concerning the truth of my propositions" since BROUNCKER had been enterprising enough to make a numerical verification. For the ratio of the circumference to the diameter (of a circle) he had found it to be

$$\left.\begin{array}{r}\text{more than } 3.14159,\ 26535,\ 69+\\ \text{less than } 3.14159,\ 26536,\ 96+\end{array}\right\}\text{ to } 1, \tag{5.1}$$

which agreed with values found by others, and in continuing the computation he obtained ratios which were alternately in excess and in defect.

No direct evidence is available concerning the method used by BROUNCKER to obtain the approximations (5.1) to the indicated accuracy. The use of WALLIS's product (1.1) or BROUNCKER's continued fraction equivalent (2.1) for this is computationally infeasible. But a possible answer to this question can be obtained by iterating a method of obtaining (4.7), the equivalent of BROUNCKER'S continued fraction approximations (4.5).

In (4.7), the term $(-1)^n/4n$ may be regarded as the sum of a geometric series approximation to $\sum\limits_{k=n}^{\infty} a_k$ where $a_k = (-1)^k/(2k+1)$. Thus

$$S_{n-1} + \frac{(-1)^n}{4n} = S_{n-1} + \frac{a_n}{1-(a_n/a_{n-1})} = S_{n-1} + \frac{a_{n-1}a_n}{a_{n-1}-a_n}. \tag{5.2}$$

From this and (4.4), there follows

$$\frac{\pi}{4} = a_0^{(1)} + \sum_{k=1}^{\infty} a_k^{(1)} = \frac{3}{4} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k(2k+1)(2k+2)}. \tag{5.3}$$

Let $S_{n-1}^{(1)}$ denote the sum of the first n terms on the right. On proceding as above, one obtains approximations to the sum of this series in the form

$$S_{n-1}^{(1)} + \frac{a_{n-1}^{(1)}a_n^{(1)}}{a_{n-1}^{(1)}-a_n^{(1)}} = S_{n-1}^{(1)} + \frac{(-1)^{n-1}}{8n(2n^2+1)} \tag{5.4}$$

whence a new series may be obtained, etc. But it is more convenient for computational purposes to rewrite the expressions in (5.2) and (5.4) in the form

$$S_n^{(i+1)} = S_n^{(i)} + \frac{(S_n^{(i)}-S_{n-1}^{(i)})\cdot(S_{n+1}^{(i)}-S_n^{(i)})}{(S_n^{(i)}-S_{n-1}^{(i)})-(S_{n+1}^{(i)}-S_n^{(i)})}; \quad i, n = 0, 1, 2, \ldots \tag{5.5}$$

where $S_{-1}^{(i)} = 0$ for all i. This is equivalent to AITKEN's δ^2 method for accelerating the convergence of sequences in numerical analysis which has been extensively developed in recent years.[33]

[33] See *e.g.*, C. BREZINSKI, "Accéleration de la Convergence en Analyse Numérique," *Lecture Notes in Mathematics*, Vol. **584** (1977).

If the procedure of (5.5) is applied to the partial sums $S_0, S_1, \ldots, S_{15}$, one finds $\pi = 3.141592653589\ldots$ and

$$4S_7^{(4)} = 3.141592653527+ < B_L < 4S_9^{(4)} = 3.141592653573+$$

$$4S_8^{(4)} = 3.141592653620+ < B_U < 4S_6^{(4)} = 3.141592653719+$$

where B_L and B_U denote BROUNCKER's lower and upper bounds in (5.1) respectively.

As mentioned above, BROUNCKER's actual method of obtaining (5.1) is unknown. But there is evidence which indicates that by about the mid-1650's, he had developed sophisticated series techniques, relative to the state of mathematical knowledge of the time. In a remarkable article published in 1668,[34] but in which the principal result had been obtained at least eleven years previously,[35] he represented the quadrature of a hyperbolic segment as the sum of an infinite series, and thus developed the first infinite series for a logarithm[36] equivalent to the series for ln 2 mentioned at the beginning of Section 4.

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[34] W. BROUNCKER, "The Squaring of the Hyperbola ...", *Philosophical Transactions of the Royal Society*, Vol. 3 (1668) 645–649. In a preface to this is a reference to a statement by WALLIS in 1657, in a monograph (see footnote 35) dedicated to BROUNCKER, concerning the latter's "quadrature of the hyperbole".

[35] J. WALLIS, *Adversus Marci Meibomii* ..., Oxford, 1657, 2–3.

[36] By a procedure which was likely suggested by ARCHIMEDES' quadrature of a parabolic segment (see footnote 24); BROUNCKER represented the area enclosed by an equilateral hyperbolic segment by an infinite series of areas of inscribed rectangles.